Exam 1.

Due Friday Feb. 12 before class. The equivalence relation F in item 1 was redefined on Monday Feb. 8.

The task is to check that the Baer classification theorem implies that the isomorphism of subgroups of $\langle \mathbb{Q}, + \rangle$ is bireducible with E_0. I supply the basic definitions and theorems and split the task into largely independent steps.

Let G be a group, $g \in G$ an element and $p \in \omega$ a prime. The p-character of g is the largest $n \in \omega$ such that the equation $p^n x = g$ has a solution in G if it exists, otherwise the p-character of g is ω. The character of g is the sequence of p-characters of g as p runs through all primes. The following theorem is taken for granted.

Theorem 0.1. (Baer 1937) *Let $G, H \subset \mathbb{Q}$ be subgroups and $g \in G, h \in H$ be nonzero elements. The following are equivalent:*

1. *G is isomorphic to H;*
2. *the character sequence of g in G is F-equivalent to the character sequence of h in H, see item 1 below for the definition of F.*

The beauty of the theorem resides in the fact that the isomorphism may not take g to h. Now, the questions for the exam:

1. Let $Y = (\omega \cup \{\omega\})^\omega$, and let F be the equivalence on Y connecting $y_0, y_1 \in Y$ if for all $n \in \omega$, $y_0(n) = \omega \leftrightarrow y_1(n) = \omega$ and for all but finitely many $n \in \omega$, $y_0(n) = y_1(n)$. Equip Y with a suitable Polish topology and prove that F is bireducible with E_0.
2. Equip the set of subgroups of $\mathbb{Q}$ with a suitable Polish topology, forming a Polish space X.
3. Find a Borel map $f \colon X \to \mathbb{Q}$ which selects from each nontrivial group its nonzero element, and prove that the map $h \colon X \to Y$ assinging each $G \in X$ the sequence of characters of $f(G)$ in G is Borel.
4. Prove that for every $y \in Y$, there is a subgroup $k(y)$ of $\mathbb{Q}$ containing 1 such that y is the sequence of characters of 1 in it. Construct the subgroup so that $k \colon Y \to X$ is a Borel map.
5. Use Baer's theorem to conclude that the isomorphism equivalence on X is bireducible with E_0.

The point of the exam is the careful checking of all details such as the Borelness of all the maps (are they continuous?)