Contents

0.1 Lecture 1 – Introduction and course objectives

0.1.1 Texts

The text will be *Classical Descriptive Set Theory* by Alexander Kechris. Alternative texts exist, but are not suitable as replacements for the Kechris text for a variety of reasons.

- *Descriptive Set Theory* by Yiannis Moschovakis[1]
- *Set Theory* by Thomas Jech [2]
- *Invariant Descriptive Set Theory* by Su Gao [3]

0.1.2 Course Outline

The course will be organized into 3 blocks, and at the end of each block there will be a take-home exam. There will be two weeks allotted for the completion of each exam[Z1].

In terms of requisite knowledge, it would be helpful to be familiar with topological spaces, groups, and measures, particularly Borel probability measures.[Z2] Otherwise there will not really be prerequisites to what I will be talking about.

0.1.3 What is Descriptive Set Theory?

DST is the study of "definable" sets in mathematical analysis. "Definable" sets in the context of DST means Borel analytic subsets of Polish spaces. We can use the Borel hierarchy to gauge the complexity of these subsets. The first task is to define the category of Polish spaces, and then persuade you that just about all of analysis takes place in Polish spaces, and that most objects of interest to mathematical analysis are in fact Polish spaces. We will show the various categories of objects that can be organized into Polish spaces.

Example 0.1.1.
An object in the category of compact metrizable spaces can be a Polish space. Let S be a compact metrizable space. We will show that

$$S \cong K([0,1]^{\mathbb{N}}),$$

where $[0,1]^{\mathbb{N}}$ is the Hilbert cube, and $K(X)$ denotes the space of all compact subsets of X equipped with the Vietoris topology. We will show that $[0,1]^{\mathbb{N}}$ is a Polish space, and that if X is Polish, $K(X)$ is Polish. Therefore we can show that every compact metrizable space S is homeomorphic to a Polish space.

Example 0.1.2.
Banach spaces will also be seen to be Polish.

We will study the category of Polish groups, which are a kind of topological group. If you have not encountered the notion of topological group before, it is a natural idea. A topological group (G, τ) is a group G together with a topology τ on G, such that the group's binary operation and inverse operation are continuous with respect to the topology.

We will study an interesting feature of DST, **dichotomies**, which are not accessible by other means.

[1] "Not as good as Kechris."
[2] Only certain sections cover DST, too general.
[3] This semester is Classical DST, Su Gao and Invariant DST comes next semester.

[Z1] "I like to give you two weekends to work on it."
[Z2] "Really you don't need to know much."

Example 0.1.3 (Sketch of a dichotomy-type theorem).
Either a Borel set is "simple", or it contains a canonical obstacle to being "simple".[Z3]

Theorem 0.1.1 (Perfect set theorem).
If $A \subseteq \mathbb{R}$ is analytic, then exactly one of these two will occur:

- *A is countable* [4]
- *A contains a non-empty perfect subset*

A perfect set is closed and has no isolated points. A basic result is that perfect sets cannot be countable. Then this theorem gives us a strict dichotomy about the cardinality of an analytic subset A. This theorem lets us show, without any additional conditions or knowledge of A, that A must either be countable, or uncountable.[5]

Another notion of simplicity could be as follows. Consider a graph $\langle X, G \rangle$ on a Borel set X. The chromatic number of a graph is the smallest number of anticliques that can cover the graph. This number is quite difficult to evaluate, even for finite graphs. We can use DST to ask what is the Borel chromatic number, that is, the smallest number of Borel anticliques.

0.1.4 Spring 2016

In the spring continuation, DST II, we will be looking at classification of analytic equivalence relations.

Example 0.1.4.
Are these two compact metrizable spaces homeomorphic, or not?

Example 0.1.5.
Are these two separable Banach spaces isomorphic, or not?

Example 0.1.6.
Are these two varieties homeomorphic, or not?

We want to know the answer to these questions, because then we can classify these objects up to the corresponding morphism. To do this, we can compare the objects by their complexity in a descriptive sense, and use this to get traction on these problems, even in a very general setting.

[4]Here, A being countable is the notion of "simplicity" referred to in Example 0.1.3.
[5]You can use this to show that e.g. all analytic sets satisfy the continuum hypothesis.

[Z3]"It is very surprising that theorems of this kind can be proven."

0.2 Lecture 2 – Topological spaces, homeomorphism, topological properties

0.2.1 Topological spaces

Definition (Topological space).
A topological space is a pair $\langle X, \tau \rangle$, where X is a set, and τ is a collection of subsets of X, satisfying the following conditions:

1. $X, \emptyset \in \tau$
2. If $\langle x_n : n \in \omega \rangle \in \tau$, then $(\bigcap_{n \in \omega} x_n) \in \tau$ (closed under countable intersection).
3. If $\langle x_n : n \in I \rangle \in \tau$, then $(\bigcup_{n \in I} x_n) \in \tau$ (closed under arbitrary union).

Definition (Continuous function).
For $\langle X, \tau \rangle$ and $\langle Y, \sigma \rangle$ topological spaces, [Z4] a function $f : X \to Y$ is called continuous if pre-images of σ-open sets are τ-open. $(\forall B \subseteq X (B \in \sigma \Rightarrow f^{-1}(B) \in \tau))$.

Definition (Homeomorphic function).
A function f is a homeomorphism if f is bijective and both f and f^{-1} are continuous. If f is a homeomorphism, the two topological spaces which f associates are called homeomorphic.

The question of whether or not two topological spaces are homeomorphic is a very basic question and arises often. To show that they are homeomorphic, one must exhibit a homeomorphism between them. To show that they are not homeomorphic, you must develop some means of proving that they are not. This may involve showing that no homeomorphism can exist, and this is usually done by finding a structure in one which cannot be continuously mapped to a structure in the other, or that such a map would not preserve a certain property, etc.

Definition (Basis for a topology).
Let $\langle X, \tau \rangle$ be a topological space. Then $A \subseteq \tau$ is a basis for τ if every set in τ is a union of some sets in A.

Definition (Sub-basis for a topology).
Let $\langle X, \tau \rangle$ be a topological space. Then $A \subseteq \tau$ is a sub-basis if τ is the smallest (relative to set-theoretic inclusion) topology containing A as a subset. Equivalently, $A \subseteq \tau$ is a sub-basis if $\{\cap A' : A' \subset A \text{ is finite}\}$ is a basis for τ.

Topologies are generally identified by identifying a sub-basis. We say that "such-and-such sets are basic open," or "these are the basic open sets," meaning that the sets which form the sub-basis of the topology we're looking at have a given form. We must often ask ourselves when working in a general setting, "what are the basic opens here?"

Example 0.2.1.
$\mathcal{P}(X)$, the discrete topology on X. The basic open sets are the singleton subsets of X.[Z5]

Example 0.2.2.
Given a linear order on X, the order topology is generated by the basic open intervals $(x, y) = \{z \in X : x < z < y\}$. $\mathbb{R}$ has this topology, but so do other sets, such as ω_2.[Z6]

Example 0.2.3.
The set $\mathbb{R}^n$ has a topology generated by open balls of positive diameter.

Example 0.2.4.
On a circle, there is a topology generated by all open intervals (arcs).

[Z4] "Topological spaces are the most general context in which continuity really makes sense."
[Z5] "Equivalently, every point is isolated."
[Z6] "This is a particularly nasty linear ordered set."

Example 0.2.5.
The set $\beta\omega$, the Stone-Ĉech compactification of ω, has a topology generated by the set of all ultrafilters containing...[Z7]

0.2.2 Properties of topological spaces

Definition (Accumulation point).
Given a topological space $\langle X, \tau \rangle$, and $S \subseteq X$, we say that $x \in X$ is an accumulation point of S if for every basic open neighborhood of x, that is, $\{\forall \mathcal{O} \in \tau : x \in \mathcal{O}\}$, $\mathcal{O}$ contains some point of S which is not x.

Claim 0.2.1.
The set of all accumulation points of S is closed.

Proof. If $x \notin acc(S)$, this is witnessed by some basic open neighborhood of x. Then $\sim acc(S) = \{\cup \mathcal{O} : S \cap \mathcal{O} = \{x\}\}$, the union of all open sets which do not contain any element of S except for one (x). The union of all these witnesses is open. Then $\sim acc(S)$ is open. Then $acc(S)$ is closed. □

Definition (Dense set).
Given a topological space $\langle X, \tau \rangle$, a set $S \subset X$ is dense (in X) if it intersects every non-empty open set, i.e. every non-empty open set contains at least one element of S.

Definition (Separable set).
Given a topological space $\langle X, \tau \rangle$, X is separable if it contains a countable dense subset.

Example 0.2.6.
The canonical example of a separable space is $\mathbb{R}$ equipped with its "usual" topology, that is, basic open sets are open intervals on the line. In this space, we can see that $\mathbb{Q}$ is a countable subset of $\mathbb{R}$, and $\mathbb{Q}$ is dense in $\mathbb{R}$, since from analysis we know that any real number can be approximated arbitrarily closely by a rational number, i.e., any real number is a limit point of some sequence of rationals. Then for any open ball around a real number, there will be some rational number within that ball, and thus $\mathbb{Q}$ is dense in $\mathbb{R}$.

This fact is more familiar than it may at first appear, as it is the entire basis for the decimal number system, or numbering systems in any base, in which a real number is written as a (possibly infinite) series of rationals:

$$a_0.a_1a_2...a_n = \sum_{i=0}^{n} \frac{a_i}{10^i} \text{ (for } a_0 \in \mathbb{Z} \text{ and } 0 \leq a_i \leq 9,\ i = \{1, 2, ..., n\})$$

Example 0.2.7.
Referring back to the examples of topological spaces and their basic open sets, we can see that the other spaces derived from $\mathbb{R}$, e.g. the circle, the line, and the plane, are all separable as well, since they corresponding subsets of $\mathbb{Q}$ can be found that are dense in those spaces.

Example 0.2.8.
We can form a topological space using ω_1, the smallest uncountable ordinal, and the order topology. In this context, ω_1 is sometimes notated $[0, \omega_1)$ to emphasize that it consists of all ordinals smaller than ω_1.

Suppose we had a countable subset $S \subset \omega_1$. Then $\sup S < \omega_1$. If $\alpha = \sup S$, then S is a subset of the closed set $[0, \alpha]$, so the closure of S is a subset of $[0, \alpha]$, which is a proper subset of ω_1. Then S is not dense in ω_1. Therefore ω_1 is not separable.

[Z7] He trails off and waves the chalk in little circles in the air instead of continuing. "The point is that basic opens can be complicated."

Definition.
A topological space $\langle X, \tau \rangle$ is called compact if every open cover of X has a finite sub-cover.

Example 0.2.9.
The closed unit interval is compact.

Partition-style proof. Assume for the sake of contradiction that the closed unit interval is not compact. Let $\mathcal{C}$ be an open cover of $[0,1]$. Divide $[0,1]$ into sub-intervals $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$. Since a union of finite subcovers would be finite, it must be the case that at least one of these sub-intervals is also not compact (i.e. has no finite subcover).

Choose one of the half-intervals which is not compact, and call it $[a_1, b_1]$. We can apply the same reasoning here, and determine that one of the halves of $[a_1, b_1]$, call it $[a_2, b_2]$, is also not compact.

Consider the nested sequence of closed intervals $\langle [a_n, b_n] \rangle_{n=1}^{\infty}$, none of which is compact, i.e., finitely coverable by our original open cover $\mathcal{C}$. Consider the "length" of the intervals in this sequence. Since we are dividing by $\frac{1}{2}$ at each step, we can write the length as

$$\lim_{n\to\infty} b_n - a_n = \lim_{n\to\infty} \frac{1}{2^n}$$

Then the diameters of these intervals decreases monotonically to 0. Then $\langle [a_n, b_n] \rangle_{n=1}^{\infty}$ is a sequence of nested, closed, bounded, and non-empty intervals. Then by the Cantor Nested Interval Theorem,

$$(\bigcap_{n=1}^{\infty} [a_n, b_n]) \neq \emptyset.$$

In fact, as an immediate corollary to that theorem, if the length of the sequence goes to 0, we have that

$$(\bigcap_{n=1}^{\infty} [a_n, b_n]) = \{p\},$$

where p is an urelement/atom, a single point.

Since $p \in [0,1]$, there must be an open interval $O \in \mathcal{C}$ with $p \in O$. Then there is a some $\epsilon > 0$ such that $(p-\epsilon, p+\epsilon) \subset O$. Let m be a positive integer such that $\frac{1}{2^n} < \epsilon$. Then since $p \in [a_m, b_n]$, we must have

$$[a_n, b_n] \subset (p-\epsilon, p+\epsilon) \subset O$$

But we have just covered $[a_m, b_m]$ with a single open set from the open cover $\mathcal{C}$. This contradicts the fact that $[a_m, b_m]$ is not finitely coverable by $\mathcal{C}$.

Then $[0,1]$ is finitely coverable by $\mathcal{C}$, and is hence compact.[Z8] □

Example 0.2.10.
The real line $\mathbb{R}$, with the usual topology (basic open sets are open intervals of the line) is not compact.

Proof. We will show that any compact subspace of $\mathbb{R}$ is bounded, and hence cannot cover $\mathbb{R}$.

[Z8] "Take the unit interval and choose some open cover. Divide it in half, and, well one of these is not compact, so you choose that one. Keep dividing, and then you get the intersection of that, and it's a singleton, so, well that has a finite open cover, and so you get the contradiction."

Let $\mathcal{C}$ be the set of all open ϵ-balls of $0 \in \mathbb{R}$. Then the union of these is $\mathbb{R}$, and so this union is an open cover for any subset S of $\mathbb{R}$.

Let $\mathcal{F} \subset \mathcal{C}$ be a finite subcover of $\mathcal{C}$ for S. The union of all intervals in $\mathcal{F}$ is the largest open ϵ-ball in $\mathcal{F}$, and since $\mathcal{F}$ covers S, it follows that $S \subset \bigcup \mathcal{F}$. Then S is bounded by the radius of that largest ball, and so every compact subspace of $\mathbb{R}$ is bounded. Since $\mathbb{R}$ is not bounded, it follows that $\mathbb{R}$ cannot be compact. □

0.2.3 Basic operations on topological spaces

Taking the subspace

Verify that τ_S is indeed a topology by verifying closure under finite intersection and arbitrary union.

For a topological space $\langle X, \tau \rangle$, and a set $S \subseteq X$, S can be turned into a subspace $\langle S, \tau_S \rangle$ of $\langle X, \tau \rangle$, by equipping it with a topology τ_S inherited from X in the following way:

$$\tau_S = \{\mathcal{O} \cap S : \mathcal{O} \in \tau\}.$$

This is called the subspace topology, relative topology, or induced topology. It is the coarsest topology on S such that the inclusion map $f : S \to X$ is continuous.

Taking the product

For an arbitrary index set I, and topological spaces $\{\langle X_i, \tau_i \rangle : i \in I\}$, if we take the (possibly infinite)[Z9] Cartesian product $\prod_{i \in I} X_i$, it turns out there is a natural topology that will turn the Cartesian product of the sets X_i into a topological space: the product topology $\prod_{i \in I} \tau_i$.

Consider a function $f : (\prod X_i) \to X_i$. This is called a canonical projection. The product topology is defined to be the coarsest topology (i.e. the one with the fewest open sets) for which every canonical projection is continuous.

The product topology is generated by sets of the form $f_i^{-1}(\mathcal{O}_i)$, where $i \in I$ and $\mathcal{O}_i$ is an open subset of X_i. These open sets are sometimes called open cylinders. Their intersections are cylinder sets. A subset of $\prod X_i$ is open if and only if it is a union of intersections of finitely many sets of the form $f_i^{-1}(\mathcal{O}_i)$.

Prove that 2^ω, ω^ω, $[0,1]^\omega$, and $\mathbb{R}^\omega$ are not homeomorphic to eachother.

Show that 2^ω is compact. Hint: because 2 is compact, product of compact is compact, etc.

Example 0.2.11 (Notable product spaces).

2^ω	(Cantor space)	Countably many copies of $\{0, 1\}$.
ω^ω	(Baire space)	Countably many copies of ω with the discrete topology.
$[0,1]^\omega$	(Hilbert cube)	Countably many copies of $[0, 1]$ with $\mathbb{R}$'s topology.
$\mathbb{R}^\omega$		Countably many copies of $\mathbb{R}$ with $\mathbb{R}$'s topology.

[Z9] If I is infinite, we may need the Axiom of Choice to show that the underlying set is non-empty.

0.3 Lecture 3 – Topological structures, metric spaces, metrizability

0.3.1 Topological structures

It is possible to put additional structure on a topological space. There are many examples, and they are commonly used in many areas of mathematics.

Example 0.3.1 (Topological group).
A topological group $\langle G, \cdot, ^{-1}, \tau\rangle$ is a group $\langle G, \cdot, ^{-1}\rangle$ together with a topology τ on G, such that the operations $\cdot$ and $^{-1}$ are continuous maps of $G \times G \to G$ and $G \to G$, respectively.

Example 0.3.2 (Topological vector space).
A topological vector space $\langle V, +, \cdot, \tau\rangle$ over $\mathbb{R}$ is a vector space $\langle V, +, \cdot\rangle$ over $\mathbb{R}$ together with a topology τ on V such that the operations $+$ and $\cdot$ are continuous maps of $V \times V \to V$ and $\mathbb{R} \times V \to V$, respectively.

In a similar way, we can define topological rings, topological fields, and so on. These structures will appear later in the course, so it is good to be aware of them now.

0.3.2 Metric spaces

All of the topological spaces studied here (in this course) are generated by a metric. What does it mean for a topology to be generated by a metric?

Definition (Metric).
Let X be a set. A function $d : X \times X \to \mathbb{R}_0^+$ is a metric if for all $x, y, z \in X$:

- $d(x, x) = 0$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$.

The topology generated by the metric d is the one generated by all open metric balls, sometimes called d-balls.

Definition (Metric topology).
Given a set X, the metric topology or the topology generated by a metric d is the one generated (whose sub-basis consists of) open metric balls

$$\{B_d(x, \epsilon) : x \in X,\ (\epsilon > 0) \in \mathbb{R}\ \},$$

where each open metric ball B_d is given by

$$B_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}.$$

Definition (Metric space).
A topological space $\langle X, \tau\rangle$ is called a metric space, sometimes notated $\langle X, d\rangle$, for d a metric, if it is a topological space generated by (induced by) the metric d.

Definition (Metrizability).
A topological space $\langle X, \tau\rangle$ is metrizable if there exists a metric d such that the topology induced by d is τ. Another way to state this is that a topological space $\langle X, \tau\rangle$ is metrizable if it is homeomorphic to a metric space.

What is the difference between general topological spaces and metric spaces? Why would we be interested in metric spaces in particular? Well, metrizable spaces have nice features.

This is a consequence of the triangle inequality.

1. Metrizable spaces are regular Hausdorff. That is, given $x \neq y \in X$, there are disjoint open sets $\mathcal{O}_x, \mathcal{O}_y$ such that $x \in \mathcal{O}_x$, $y \in \mathcal{O}_y$, and $\mathcal{O}_x \bigcap \mathcal{O}_y = \emptyset$.
2. Every point in a metrizable space has a countable basis of neighborhoods, its "neighborhood basis", i.e. $\{B(x, \epsilon) : \epsilon > 0,\ \epsilon \in \mathbb{Q}\}$.
3. There are many others

0.3.3 Metrizability theorems

These are theorems that will say "if you have a topological structure, there is a nice metric which can generate that topology."[Z10]

Theorem 0.3.1 (Urysohn Theorem).
For a topological space $\langle X, \tau \rangle$, the following are equivalent

1. *X is separable metrizable*
2. *X has a countable basis and is regular Hausdorff*

There will be a similar theorem for topological groups. Given some $\langle G, \cdot, ^{-1}, \tau \rangle$, you may want a metric d such that d generates τ, and $d(xy, xz) = d(y, z)$. This would be a metric which is invariant from the left. And you can do that from the right as well.

Proposition 0.3.1.
Let X be a set, τ a topology, and d a metric. Then d generates τ if and only if the following are satisfied

1. $\forall \mathcal{O} \in \tau\ \forall x \in \mathcal{O}\ \exists \epsilon > 0$ *such that* $B(x, \epsilon) \in \mathcal{O}$.
2. $\forall x \in X\ \forall \epsilon > 0\ \exists \mathcal{O} \in \tau$ *such that* $x \in \mathcal{O} \subseteq B(x, \epsilon)$.

Proof. To show that every open metric ball belongs to τ, consider $B(x, \epsilon)$. Given $y \in B(x, \epsilon)$, there is some $\delta > 0$ such that $B(y, \delta) \subseteq B(x, \epsilon)$. By condition 2, there will be some open set $\mathcal{O} \in \tau$ such that $y \in \mathcal{O}$ and $\mathcal{O} \subseteq B(x, \epsilon)$.

Then show that the union of every open ball containing a point in $B(x, \epsilon)$ is $B(x, \epsilon)$. □

This needs to be checked and probably fixed.

0.3.4 Metrizable spaces

Example 0.3.3 ($\mathbb{R}$).
The topology of the real numbers $\mathbb{R}$ can be generated by the Euclidean metric, the absolute value of the difference between two points. $d(x, y) = \sqrt{(x - y)^2} = |x - y|$.

Example 0.3.4 ($\mathbb{R}^n$).
In general, the topology of $\mathbb{R}^n$ can be generated by the Euclidean distance, generalized a bit. For points $a, b \in \mathbb{R}^n$ such that $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$, we have $d(a, b) = \sqrt{\sum_{i=1}^{n} (b_i - a_i)^2}$.

There is nothing that says the metrization has to be unique.[Z11] The topology of $\mathbb{R}^n$ can also be generated by the Manhattan distance, where for points $a, b \in \mathbb{R}^n$ such that $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$, we have $d(a, b) = \sum_{i=1}^{n} |a_i - b_i|$.

Example 0.3.5 (2^ω).
Cantor space is generated by the smallest difference metric, that is $d(x, y) = 2^{-\delta(x,y)}$, where $\delta(x, y) = min\{n : x(n) \neq y(n)\}$. I claim that the product topology is generated by this metric.

[Z10] "It is always interesting when you see a topological structure to wonder whether the topology is generated by a metric."

[Z11] "This is important because if a given topological space is metrizable, there is not necessarily one "canonical" choice of metric."

Proof. We can use the proposition above. Given a metric ball around a point x, we must show that there is some open set which is a subset of the ball, and which contains x. An open ball in this topological space is an initial segment of a binary sequence.

Then since each initial segment is finite, there are a finite number of intersections of the d-balls, since δ will only pick out some finite position. □

Clean this up, it is pretty slick when he does it.

We could also generate this topology by finding closed subsets of $\mathbb{R}^n$ which are homeomorphic to 2^ω and apply the Euclidean metric.

For example, use Cantor's middle third set. This is homeomorphic to 2^ω, and is a closed subset of $\mathbb{R}$. Or we could use "Sierpinski's carpet".[6]

And there are many others.

Example 0.3.6 (ω^ω).
Can also be metrized with the smallest distance metric.

0.3.5 Non-metrizable spaces

Example 0.3.7 ($\mathbb{R}^\mathbb{R}$).
The set $\mathbb{R}^\mathbb{R}$ of all functions $f : \mathbb{R} \to \mathbb{R}$ is not metrizable. They have a topology of pointwise convergence, which is really the product topology. Here, points do not have countable bases, so the space is not metrizable.

Think about why $\mathbb{R}^\mathbb{R}$ is different from $\mathbb{R}$.

Example 0.3.8 ($\beta\omega$).
The set $\beta\omega$, the Stone-Ĉzech compactification of ω, is not metrizable, for reasons that "get complicated".

0.3.6 Completely metrizable spaces

Definition (Complete metric).
A metric d is called complete if any Cauchy sequence converges (has a limit). That is, if for all $\langle x_n : n \in \omega \rangle$, for all $\epsilon > 0$, there is some n such that for all m and k greater than n, $d(x_m, x_k) \leq \epsilon$.

Mostly we are interested in complete metrics, because for example in differential equations, and approximation, you want to look at accumulation points in a space.

Definition (Completely metrizable space).
A space is completely metrizable if there is a complete metric generating its topology.

As an example of complete versus non-complete metrics, consider the set ω with the discrete toplogy. If we metrize it with the discrete metric, we get a complete metric. If we metrize it by setting, say $d(m, n) = \frac{1}{m} - \frac{1}{n}$ if $m < n$, then the metric is not complete.

However, in both examples, we have successfully generated the discrete topology. Therefore it might be possible to metrize a space in both a complete and a non-complete way.

The take-home is that we have several different metrics here, but only one topology. So we think of topology over metrics.

Definition (Polish space).
A Polish space is a separable, completely metrizable space.

[6] He hesitated a few times, something sounded wrong to him about that name, and he mentioned that maybe it is called something else. But he was correct.

0.4 Lecture 4 – Polish spaces

0.4.1 Examples of Polish spaces

Proposition 0.4.1.
In a metrizable space, closed sets are G_δ. A G_δ set is a countable intersection of open sets $\bigcap_{n\in\omega}\mathcal{O}_n$ for some open sets $\mathcal{O}_n$.

Proof. Let $\langle X, d\rangle$ be the metric space, and $\mathcal{C} \subseteq X$ a closed set. Let $\mathcal{O}_n = \{x \in X : \exists y \in \mathcal{C} : d(x,y) < 2^{-n}\}$. Observe that each $\mathcal{O}_n$ is open. Observe that $\mathcal{C} = \bigcap_{n\in\omega}\mathcal{O}_n$. So if $x \in \mathcal{C}$, then for all n, $x \in \mathcal{O}_n$. If $x \in \bigcap_{n\in\omega}\mathcal{O}_n$, there exist points $y_n \in \mathcal{C}$ such that $d(x, y_n) < 2^{-n}$. Then the sequence $\langle y_n : n \in \omega\rangle$ converges to $x \in \mathcal{C}$. □

Example 0.4.1.
The following are Polish spaces.

- Cantor space
- Baire space
- Euclidean space
- Separable Banach spaces

0.4.2 Operations on Polish Spaces

Taking a G_δ subspace

There will be a theorem soon where we will show that a subset of Polish space is Polish if and only if the subset is G_δ.

Taking a countably infinite product

We must preserve countability or we will not maintain separability.

Spaces of continuous functions

If X is compact[Z12], and Y is Polish, then $C(X,Y)$, that is, the space of all continuous functions from X to Y, is Polish.

If X and Y are $\mathbb{R}$ or $\mathbb{C}$, then $C(X,Y)$ is an easy example of a Banach space.

Hyperspaces

If X is Polish, one can form $K(X)$, the space of compact non-empty subsets of X. One can also form $F(X)$, the space of non-empty closed subsets of X. These are called hyperspaces.[Z13]

Space of all Borel probability measures on X

Sometimes this is notated $M(X)$, for a Polish space X. This is the beginning of dynamical systems, where you look at self-homeomorphisms and things of that nature.

[Z12] "What if X is not compact? Well, we can look at the space of continuous functions with compact support, or whose limit at infinity is 0, and so on..."
[Z13] "It sounds like something very fancy."

More structural operations

If G is a Polish group, and $H \subset G$ is a closed subgroup, then we can form G/H. It turns out that the cosets here, G/H, can be equipped with a topology that makes them Polish.

0.4.3 Digression: metrizable spaces

We will need a few more tools for the coming proofs. If X is a metrizable space with a metric d, can you mess[7] with d to get the same topology? Yes. there are a few ways to do this.

- Cut d off at any positive constant. That is, define $e(x, y) = d(x, y)$ if $d(x, y) \leq \epsilon$ for some ϵ, and $e(x, y) = \epsilon$ otherwise. Then you can verify that e is a metric, and generates the same topology.
- We can generalize the previous item and apply any concave increasing continuous function f to d, if $f(0) = 0$. Examples would be $\sqrt{d}$, $log(d)$, etc. You should verify the triangle inequality for these to see they are still metrics.
- If $f : X \to \mathbb{R}$ is a continuous function, then define $e(x, y) = d(x, y) + |f(x) - f(y)|$. This gives you a different metric, it is like the sum of two metrics, really. It generates the same topology.
- If d_n is a countable collection of metrics generating the same topology, such that $|d_n| < 2^{-n}$, then $\sum d_n$ is again a metric, and generates the same topology.
  To see this, let m be so large that $\sum_{m \geq n} 2^{-m} \leq \frac{\epsilon}{2}$. For each $n < m$, look at $B_n(x, \frac{\epsilon}{2n})$. Let $\mathcal{O} = \bigcap_{n<m} B_n(x, \frac{\epsilon}{2n})$. Then $\mathcal{O} \subset B_e(x, \epsilon)$.

To verify, you must check that for any d-ball, you can find an e-ball which is a subset of the d-ball, with the same center, and vice-versa.

Verifying this uses the fact that f is continuous.

0.4.4 Elementary results about Polish spaces

Theorem 0.4.1.
Let X be Polish, and $B \subset X$ a set. Then the following are equivalent.

1. *B is Polish with the restricted topology.*
2. *B is G_δ in X.*

Proof. $(2 \to 1)$ Assume that B is open. Clearly B is separable. But we don't know whether it is metrizable. The restriction of B to this topology generates the topology, but the metric is not complete.

Let d be the metric on X. Let $f : B \to \mathbb{R}$ be the function which is equal to the distance to the complement. (Since the complement is closed, we can look at the distance to the infimum of the complement).

f is a continuous nonzero function on B (it is nonzero because B is open).

Then look at the metric $e(x, y) = d(x, y) + |\frac{1}{f(x)} - \frac{1}{f(y)}|$. This generates the same topology, but now, it is complete. How do we know it is complete? Because as you start getting close to the border, the sequence stops being Cauchy.

If B is G_δ, then $B = \bigcap_{n\in\omega} \mathcal{O}_n$. So for each $\mathcal{O}_n$ there is a complete metric. Fix d_n at each n. Choose $d_n \leq 2^{-n}$ on each $\mathcal{O}_n$. Let d be the sum metric of the intersection. Check for completeness.

(If Cauchy in d, then Cauchy in d_n). □

The lecture ended here. Tune in next time for the conclusion of the proof.

[7] His words

0.5 Lecture 5 – Results about Polish spaces

0.5.1 Elementary results about Polish spaces (continued)

We are going to just re-prove both directions of the theorem that we didn't finish last time, even though we finished one direction last time.

Theorem 0.5.1.
Let X be Polish, and $B \subset X$ a set. Then the following are equivalent.

1. *B is Polish with the restricted topology.*
2. *B is G_δ in X.*

Proof.
$(2 \to 1)$
Let d be a complete metric on X. Let $e(x,y) = d(x,y) + |\frac{1}{dist(x,\sim B)} - \frac{1}{dist(y,\sim B)}|$. Then e is a metric, as we observed last time. But we must check for completeness.

Suppose $\langle x_n : n \in \omega \rangle$ is an e-Cauchy sequence. Since e is a sum of positive numbers, each must form a d-Cauchy sequence. So it has a d-limit, call it y. Why is $y \in B$ true? Look at $\frac{1}{dist(x,\sim B)}$. This is also a Cauchy sequence. So there must be a limit, call it r. Then $dist(y, \sim B) = \frac{1}{r}$. Then $y \notin \sim B$, and so $y \in B$.

To conclude, note that a subspace of a separable metrizable space is separable metrizable. $X \supseteq \{x_n : n \in \omega\}$, and $B \subseteq X$. If X is metrizable, let $\{y_{mn} : m, n \in \omega\}$ be points such that $d(x_n, y_m n) < \frac{1}{m}$, and $y_{mn} \in B$, if possible (there may not be any such). Argue that this set must be dense in B.

$(1 \to 2)$
Suppose first that B is dense. We will see at the end that this is a reasonable and harmless assumption. Let e be a computable metric on B generating its topology. For each $n \in \omega$, let $\mathcal{O}_n = \{x \in X : \exists P$ an open nbhd of x s.t. $diam_e(P \cap B) < \frac{1}{n}\}$.

We fix n and say that x is in a neighborhood if the intersection has small diameter. Observe

- $\mathcal{O}_n$ is open ($\mathcal{O}_n \subseteq X$). Why? It is the union of open sets.
- Show $B = \cap \mathcal{O}_n$. Because $x \in B$, $B(x, \frac{1}{2n})$, there is an open set in the topology of X.

For every $x \in B$, find an open set $\mathcal{O} \subseteq X$ such that $(\mathcal{O} \cap B) = B_e(x, \frac{1}{3n})$ (B_e is a subset of B with e-diameter less than $\frac{1}{n}$, so $\mathcal{O}$ witnesses $x \in \mathcal{O}_n$). So $B \subseteq \mathcal{O}_n$.

Then $(\cap \mathcal{O}_n) \subseteq B$. Why? Let $x \in \bigcap_{n \in \omega} \mathcal{O}_n$, and let P_n be neighborhoods witnessing $x \in \mathcal{O}_n$. Now find $x_n \in B \cap (\bigcap_{m \in n} P_m)$. P_m is an open set of some sort and B is dense, so $B \cap (\bigcap_{m \in n} P_m)$ is not empty.

Now observe that the sequence $\langle x_n : n \in \omega \rangle$ is Cauchy in e. The radius of P_m is getting smaller, so the sequence must be Cauchy. So since e is complete, there must be a limit.

The limit must be in B, because e is complete on B. We want to argue that the limit is x.

To show the limit is x, choose $B_d(x, \frac{1}{n})$ for some metric d. Then $x_n \in B \cap (\bigcap_{m \in n} P_m \bigcap B_d(x, \frac{1}{n}))$.

If B is not dense in X then let Y be the closure of B. B is not dense, but Y is an arbitrary Polish topological space, so we will look at $Y = \bar{B}$. Then since it is closed, it is G_δ, and so it is Polish. Then B is G_δ in Y, so $B = Y \cap \mathcal{O}_n$ for some G_δ subset of X. But Y is closed, so Y is G_δ. So B is G_δ.

□

Theorem 0.5.2.
If $\langle X_n : n \in \omega \rangle$ are Polish spaces, then so is the product $\prod_{n\in\omega} X_n$, when equipped with the product topology.

Proof.
(Completely metrizable)
If d_n are complete metrics on each X_n, let $d(x, y) = \sum_n d_n(x(n), y(n))$ be a metric on the product. The problem is, the product could be infinite. So we cut it off, so that $|d_n| < 2^{-n}$. Then check for completeness and that it generates the product topology.

(Separable)
Let $\langle Y_n : n \in \omega \rangle$ be countable dense sets in X_n. Let $Y = \prod_{n\in\omega} Y_n$. But the product of countably many infinite sets is uncountable. Dense, sure, but uncountable. So let $Y = \{x \in X :$ all but finitely many $n \in \omega$, $x(n) = z_n \in X$, otherwise $x(n) \in Y_n$ $\}$.

Why is this set dense? Y is countable, since every Y is given by a finite set of ω. So it is countable. Then why dense? (visual argument on board ensues). □

Theorem 0.5.3.
Every Polish space is homeomorphic to a G_δ subset of the Hilbert cube $[0, 1]^\omega$.

Because of this last theorem, the operations of countable product and taking G_δ subspace generate every Polish space that there is. Still, the others are worth studying and have their own interesting effects, but it is worth remembering.

0.6 Lecture 6 – Compact spaces

0.6.1 Review

Let's pause to see how our powers have grown just from the last two results. Here are some examples of new results accessible to us using the theorems we proved in Lecture 5.

Example 0.6.1.
Look at the irrationals $\mathbb{R} \setminus \mathbb{Q}$ with the topology inherited from $\mathbb{R}$. Descriptively, $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ set. Why? Well, $\sim (\mathbb{R} \setminus \mathbb{Q}) = \mathbb{Q}$, which is a countable set. So

$$(\mathbb{R} \setminus \mathbb{Q}) = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\}.$$

Then $\mathbb{R} \setminus \mathbb{Q}$ is Polish, by Theorem 0.5.1, and homeomorphic to Baire space.

Example 0.6.2.
We can easily show using Theorem 0.5.2 that $[0,1]^\omega$ is Polish. We can show also that a set is Polish if and only if it is homeomorphic to a G_δ set of $[0,1]^\omega$.

0.6.2 Compact spaces

Before we prove the next theorem, let's stop and review some basic observations about compact spaces that will help us in the next few results.

1. A closed subset of a compact space is compact.
2. Every continuous function from a compact space to $\mathbb{R}$ attains its supremum.[Z14]
3. Every sequence in a metrizable compact space has a convergent subsequence.[Z15]

0.6.3 Another result about Polish spaces

Theorem 0.6.1.
If X is a compact Polish space, and Y is a Polish space, then $C(X, Y)$ is Polish.

Proof. Let d be a complete metric on Y. We will argue that for $f, g \in C(X, Y)$, the metric $e(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$ is complete, and generates the compact-open topology on $C(X, Y)$. We then prove that $C(X, Y)$ is separable, and thus Polish.

Claim 0.6.1.
The metric e is complete.

Proof of claim. Suppose $\langle f_n : n \in \omega \rangle$ is an e-Cauchy sequence. For each $x \in X$, the sequence $\langle f_n(x) \rangle$ is d-Cauchy, because the distance between them is less than the distance d. Since d is complete, $\langle f_n(x) : n \in \omega \rangle$ converges to some $g(x)$.

Now we must argue that g is the limit. To do this, we must prove that

1. g is a continuous function from $X \to Y$.
2. $e(f_n, g)$ tends to zero.

[Z14] "This statement contains an implication that the set is bounded."
[Z15] "It must be metrizable, since the Stone-Czech ω is compact, but has no compact subsequence."

We prove the second, and leave the first to the reader. Observe that if $\langle y_n : n \in \omega \rangle$ is a converging sequence of points in Y with $diam(\langle y_n : n \in \omega \rangle) \leq \epsilon$, then clearly $d(z, y_0) \leq \epsilon$.

Given some $\epsilon > 0$, we must find some $n_\epsilon \in \omega$ such that $\forall n > n_\epsilon$, $e(f_n, g) \leq \epsilon$. Which is the same as finding $\sup\{d(f_n(x), g(x) : n \in \omega,\ x \in X\}$.

So if we find such a thing, then $\forall x \in X$, $d(f_{n_0}(x), f_{n_1}(x)) \leq \epsilon$. Then for each $x \in X$ and each $m > n_\epsilon$, $d(f_n(x), g(x)) \leq \epsilon$. And then $\forall n > n_\epsilon$, $e(g(x), f_n(x)) \leq \epsilon$. ■

We are chasing around definitions to get what we want.

Claim 0.6.2.
The metric e generates the compact-open topology

Proof of claim. Now we must show that e generates the compact-open topology. To do this, we have to again prove two separate results.

1. If $\mathcal{O} \subset C(X, Y)$ is a basic open set, and $f \in \mathcal{O}$, we need an $\epsilon > 0$ such that the e-ball around f of radius ϵ is a subset of $\mathcal{O}$.
2. If $f \in C(X, Y)$, and $\epsilon > 0$, we need a basic open set $\mathcal{O} \subset C(X, Y)$ such that $f \in \mathcal{O}$ and $\mathcal{O} \subset B_e(f, \epsilon)$.

To prove the first, find $n \in \omega$ such that $\langle K_m : m \in n \rangle$ are compact subsets of X, and $\langle \mathcal{O}_m : m \in n \rangle$ are open subsets of Y.

Then $\mathcal{O} = \{g : \forall m < n \forall x \in K_m(g(x) \in \mathcal{O}_n)\}$ is our basic open set.

We claim that $\exists \delta > 0$ such that $\forall x \in K_0$, the $B_d(f(x), \delta) \subseteq \mathcal{O}_0$. We must show this using compactness.

Consider a function $h(x) = \frac{1}{d(f(x), \sim\mathcal{O}_0)}$. Notice that h is a continuous function from $K_0 \to \mathbb{R}$, since K_0 is compact. So there is a bound r. Let $\delta = \frac{1}{r}$. Now let $\epsilon = min\{d_m : m \in n\}$.

Lecture 6 ended here, the proof of the second result and the remainder of the proof of the theorem is from the start of lecture 7.

This proves the first result.

To prove the second result, let $\mathcal{C} = \{B_d(y, \frac{\epsilon}{4}) : y \in Y\}$ be the set of all open d-balls of radius $\frac{\epsilon}{4}$i in the space Y.

Let $\mathcal{D} = \{f^{-1}(B) : B \in \mathcal{C}\}$, the set of all f-preimages of balls in $\mathcal{C}$.

Note that $\bigcup \mathcal{C} = Y$ and $\bigcup \mathcal{D} = X$, so $\mathcal{D}$ is an open cover of X. Therefore by compactness of X, there is a finite subcover $\mathcal{D}' \subseteq \mathcal{D}$ (this is why we need compactness, if we didn't assume compactness, we couldn't make this move).

We can see that f has to look like the figure. It must pass via a tunnel of width $\frac{\epsilon}{4}$. Then $\mathcal{O} = \{g : \forall i \in n \forall x \in \overline{U_i}, g(x) \in B_d(y_i, \frac{\epsilon}{2}$[Z16]$)\}$.

Note these $\mathcal{O}$ are our generated open sets, from a function from compact X to Y.

Important: $F \in \mathcal{O}$ (show why). Look at $\mathcal{O}$. $\mathcal{O} \subseteq B_e(f, \epsilon)$ Why?

This proves the second result, and hence the claim. ■

We have thus established that if d is a complete metric on Y, and $f, g \in C(X, Y)$, then $e(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$ is a complete metric, and generates the compact-open topology on $C(X, Y)$.

Finally, we have prove that $C(X, Y)$ is separable.

Remark.
Recall Weierstraß theorem: if X is compact, look at $C(X, \mathbb{R})$ or $C(X, \mathbb{C})$ with the uniform-convergence topology.

[Z16]"epsilon over... two, probably

These spaces, as opposed to $C(X, Y)$, have an algebraic structure. Really, they are an algebra, because of $\mathbb{R}$ and $\mathbb{C}$.

That is, whenever $A \subseteq C(X, \mathbb{R})$, closed under algebraic operations, is a subalgebra which separates points (that is, distinct points $x, x' \in X \Rightarrow f(x) \neq f(x')$ for a function $f \in C(X, \mathbb{R})$), then A is dense.

For example, think of a countable algebra: polynomials with variable coefficients. Clearly this separates points. But there are many others. Weierstraß used algebraic properties to show density.

In Fourier analysis, every continuous function can be approximated by a trigonometric function.

We will prove a weaker theorem, for arbitrary Polish spaces, with no structure on Y.

Claim 0.6.3.
The space $C(X, Y)$ is separable.

Proof of claim. We will prove $C(X, Y)$ is separable by obtaining a countable dense subset.

We require metrics on X and Y. Let d_X be a metric on X, and d_Y a metric on Y. Since X and Y are separable, let $D_X \subseteq X$ and $D_Y \subseteq Y$ be countable dense sets.

For every $a \subset D_X$ finite and $g : a \to D_Y$, for any rational $\epsilon > 0$ and $\delta > 0$, pick a function $f_{a,g,\epsilon,\delta} \in C(X, Y)$ such that for every $x \in X$ and for every $z \in a$, if $d_X(x, z) < \delta$, then $d_Y(f_{a,g,\epsilon,\delta}(x), g(x)) < \epsilon$, if such a function exists (it may not, and that is fine).

Note there are only countably many subsets a, functions $f_{a,g,\epsilon,\delta}$, and choices for ϵ and δ. So the set $\{f_{a,g,\epsilon,\delta}\}$ is countable, as there are only countably many choices for its parameters.

Let $E = \{f_{a,g,\epsilon,\delta}\}$. We claim E is dense.

First appearance of the enemy

Why? An enemy gives you f, and $\epsilon > 0$. I need an element of E within ϵ distance of f. We will again use compactness of X to argue that f is uniformly continuous, that is, if inputs are within δ of eachother, the outputs are within $\epsilon/2$ of eachother.

Find $\delta > 0$ such that $d_X(x_0, x_1) < \delta \Rightarrow d_Y(f(x_0), f(x_1)) < \epsilon/2$.

X is compact, so you can find a finite δ-network in D_X (by a compactness argument) and look at how f acts on this network.

Then values may not be in D_Y, but just choose any element $\epsilon/4$ close to each.

Is there a function such that $\forall x \in X \forall z \in a$ if $d_X(x, z) < \delta \Rightarrow d_Y(f(x), f(z)) < \epsilon/2$? Yes. The function that the enemy gave you. Now, argue using uniform continuity that there will be an element of E at that point.[Z17] ■

□

[Z17] "This somehow[sic] closes the subject of function spaces"

0.7 Lecture 7 – Compact spaces (continued)

0.7.1 From last time

Remark.
Should have mentioned from last time. The topology generated by the metric $e(f,g) = \sup\{d(f(x),g(x)) : x \in X\}$ is also known as the uniform-convergence topology.

Why is this? Look at $e(f,g) - \max\{f,g\}$. This says that a sequence of functions converges to a function. There is a similar topological point here. Topologies are identified by a basis or a subbasis, or by a sequence which is declared to be convergent, by saying "the topology of this convergence is the smallest topology in which these sequences are convergent."

Then we can define compact-open using convergence of sequences of functions.

When defining the compact-open topology, we seem to be using nothing about the sets X, Y. When using the uniform-convergence topology, we seem to be using the metric d on Y. But at least in a Polish context, they generate the same topology.[Z18]

0.7.2 Other Polish function spaces

There are many other function spaces that are Polish.

[Z18] "This type of invariance is... highly prized."