Definition 0.1. A *graph* is a pair $\langle V, E\rangle$ where V is a set and E is a set of pairs from V. An *anticlique* in the graph is a set $A \subset V$ such that $[A]^2 \cap E = 0$. A *chromatic number* of the graph is the smallest possible number of anticliques which together cover V. If in addition the set V is equipped with a topology, the *Borel chromatic number* of the graph is the smallest possible number of Borel anticliques which cover V.

Definition 0.2. Let $\{s_n : n \in \omega\}$ be any collection of finite binary strings such that $s_n \in 2^n$ and for every $t \in 2^{<\omega}$ there is $n \in \omega$ such that $t \subset s_n$. The graph G_0 on 2^ω is the set of all pairs of the form $\{s_n^\frown 0^\frown z, s_n^\frown 1^\frown z\}$ such that $n \in \omega$ and $z \in 2^\omega$.

Theorem 0.3. *The chromatic number of G_0 is 2 while the Borel chromatic number of G_0 is uncountable.*

Proof. For the evaluation of the chromatic number of G_0, let E_0 be the equivalence relation on 2^ω connecting binary sequences $y, z \in 2^\omega$ just in case they differ on only finitely many entries. Use the axiom of choice to find a set $A \subset 2^\omega$ which visits each E_0 class in exactly one point. Let $B_0 = \{y \in 2^\omega :$ writing z for the unique element of $A \cap [y]_{E_0}$, y and z differ at even number of entries$\}$ and $B_1 = \{y \in 2^\omega :$ writing z for the unique element of $A \cap [y]_{E_0}$, y and z differ at odd number of entries$\}$. Check that B_0, B_1 are G_0-anticliques. Clearly, $2^\omega = B_0 \cup B_1$ and so the chromatic number of G_0 is 2.

The evaluation of the Borel chromatic number uses a small claim:

Claim 0.4. *If $B \subset 2^\omega$ is a Borel non-meager set, then B contains a G_0 edge.*

Proof. The Borel set B is modulo the meager ideal equal to an open set. Since the set B is nonmeager, this means that there is a finite binary string $t \in 2^{<\omega}$ such that $B \cap [t]$ is comeager in $[t]$; let $\{O_m : m \in \omega\}$ be some countable collection of sets open dense in $[t]$ such that $\bigcap_m O_m \subset B$. Let $n \in \omega$ be such that $t \subset s_n$. By induction on $m \in \omega$ find binary strings $t_m \in 2^{<\omega}$ so that $0 = t_0 \subset t_1 \subset t_2 \subset \dots$ and both sets $[s_n^\frown 0^\frown t_{m+1}]$ and $[s_n^\frown 1^\frown t_{m+1}]$ are subsets of O_m. This is easily possible using the density of the set O_m. In the end, let $z = \bigcup_m t_m \in 2^\omega$ and $x = s_n^\frown 0^\frown z$ and $y = s_n^\frown 1^\frown z$. Both points x, y belong to $\bigcap_m O_m$ and therefore to B, and they form a G_0-edge. $\square$

As a result, every Borel G_0-anticlique is meager, and so by the Baire category theorem, countably many Borel G_0-anticliques cannot cover 2^ω. $\square$

Theorem 0.5. *Suppose that G is an analytic graph on a Polish space X. Then exactly one of the following happens:*

1. *the Borel chromatic number of G is countable;*
2. *there is a continuous homomorphism of G_0 to G.*

Here, a continuous homomorphism of G_0 to G is a continuous map h from 2^ω to X such that for any $y_0, y_1 \in 2^\omega$, $y_0\ G_0\ y_1$ implies $h(y_0)\ G\ h(y_1)$. Note the important difference between a homomorphism and reduction: in the reduction definitions, one finds an equivalence, in the homomorphism definitions, one finds an implication.

Proof. To see that (1,2) are mutually exclusive, suppose for contradiction that $h\colon 2^\omega \to X$ is a continuous homomorphism of G_0 to G and $X = \bigcup_n B_n$ is a countable union of Borel G-anticliques. Then, for each $n \in \omega$, $h^{-1}B_n$ is a Borel G_0-anticlique, and $\bigcup_n h^{-1}B_n = 2^\omega$. This, however, is impossible in view of Theorem 0.3.

To show that at least one of (1,2) must occur, start with a general preliminary claim:

Claim 0.6. *Whenever $A \subset X$ is an analytic G-anticlique, then there is a Borel anticlique $B \subset X$ such that $A \subset B$.*

Proof. Let $A_0 = \{x \in X : \exists y \in A\ \{x, y\} \in G\}$. The set $A_0 \subset X$ is analytic and disjoint from A, so by the Lusin separation theorem, there is a Borel set $B_0 \subset X$ containing A as a subset and disjoint from A_0. Let $A_1 = \{x \in X : \exists y \in B_0\ \{x, y\} \in G\}$. Againg, this is an analytic set disjoint from A and so there is a Borel set $B_1 \subset X$ containing A as a subset and disjoint from A_1. Let $B = B_0 \cap B_1$ and check that B works. □

The plan now is the following: attempt to build the homomorphism as in (2) with "finite approximations"; if this attempt fails, extract from the failure countably many Borel G-anticliques covering X. We will need some notation. Fix a continuous surjection $f\colon \omega^\omega \to X$ and another continuous surjection $g\colon \omega^\omega \to G$.

Say that p is an *approximation* if $p = \langle a_p, b_p\rangle$ so that for some $n = n_p \in \omega$, a_p is a function from 2^n to ω^n and b_p is a function such that $\mathrm{dom}(b_p) = \bigcup_{k\in n} 2^{n-k-1}$ and $\mathrm{rng}(b_p) \subset \omega^n$.

If p, q are approximations, write $q \leq p$ if $n_p \leq n_q$, for every $t \in 2^{n_q}$ $a_q(t) \restriction n_p = a_p(t \restriction n_p)$, and for every $k \in n_p$ and every $t \in 2^{n_q-k-1}$, $b_q(t) \restriction n_p = b_p(t \restriction n_p - k - 1)$.

If $p = \langle a_p, b_p\rangle$ is an approximation, then a *validation* of p is a pair $\langle \bar{a}_p, \bar{b}_p\rangle$ such that

- $\mathrm{dom}(\bar{a}_p) = \mathrm{dom}(a_p)$, $\mathrm{rng}(\bar{a}_p) \subset \omega^\omega$ and for every $t \in \mathrm{dom}(a_p)$ $a_p(t) \subset \bar{a}_p(t)$;
- $\mathrm{dom}(\bar{b}_p) = \mathrm{dom}(b_p)$, $\mathrm{rng}(\bar{b}_p) \subset \omega^\omega$ and for every $t \in \mathrm{dom}(b_p)$ $b_p(t) \subset \bar{b}(t)$;
- whenever $k \in n$ is a number and $t \in 2^{n_p-k-1}$ is a binary string, then $g(\bar{b}_p(t)) = \langle f(\bar{a}_p(s_k^\frown 0^\frown t)), f(\bar{a}_p(s_k^\frown 1^\frown t))\rangle$.

Claim 0.7. *Whenever $\langle p_n : n \in \omega\rangle$ is a descending sequence of approximations such that $n_{p_n} = n$ and each p_n has a validation, then there is a continuous homomorphism from G_0 to G.*

Proof. Just define the function $h\colon 2^\omega \to X$ by $h(y) = f(\bigcup_n a_{p_n}(y \restriction n))$. It must be proved that for $y_0, y_1 \in 2^\omega$, if $y_0\ G_0\ y_1$ then $h(y_0)\ G\ h(y_1)$. To see this, suppose that $y_0\ G_0\ y_1$ holds and find $k \in \omega$ and $z \in 2^\omega$ such that $y_0 = s_k^\frown 0^\frown z$ and $y_1 = s_k^\frown 1^\frown z$, and let $w = \bigcup_{n>k} b_{p_n}(z \restriction n-k-1)$. It will be enough to show that $g(w) = \langle h(y_0), h(y_1)\rangle$.

if this failed, then by the continuity of the functions g, f there would have to be a number n such that for no elements $\bar{y}_0, \bar{y}_1, \bar{w}$ which agree with y_0, y_1, w respectively on the first n entries, $g(\bar{w}) = \langle h(\bar{y}_0, \bar{y}_1)$. This in turn means that the approximation p_n has no validation, a contradiction. □

Thus, we are really seeking a decreasing sequence of validated approximations. The difficulty is that while p may be validated, it may still occur that none of its one step extensions are, and we have to avoid such dead ends. To do that, for a set $Y \subset X$ call an approximation p Y*-terminal* if none of its one-step extensions $q \leq p$, $q = \langle a_q, b_q\rangle$ has a validation with $\mathrm{rng}(f \circ \bar{a}_q) \subset Y$.

Claim 0.8. *Suppose that p is an approximation and $Y \subset X$ is a Borel set. If all one step extensions of p are Y-terminal, then there are countably many Borel G-anticliques $\{B_n\colon n \in \omega\}$ such that p is $Y \setminus \bigcup_n B_n$-terminal.*

Proof. Let $q \leq p$ be a one step extension of p. Write $A(q, Y) = \{f(\bar{a}_q(s_{n_q})\colon \langle \bar{a}_q, \bar{b}_q\rangle$ is a validation of q such that $\mathrm{rng}(f \circ \bar{a}_q) \subset Y\} \subset Y$. There are two cases.
Case 1. Suppose first that $A(q, Y)$ contains a G-edge for some $q \leq p$. Then q is not terminal: there must be two validations $\langle \bar{a}^0_q, \bar{b}^0_q\rangle$ and $\langle \bar{a}^1_q, \bar{b}^1_q\rangle$ of q such that $\mathrm{rng}(f \circ \bar{a}^0_q)$ and $\mathrm{rng}(f \circ \bar{a}^1_q)$ are both subsets of Y and $f(\bar{a}^0_q(s_{n_q})$ is G-connected to $f(\bar{a}^1_q(s_{n_q})$. Define the functions $\bar{a}_r, \bar{b}_r$ as follows: $\mathrm{dom}(\bar{a}_r) = 2^{n_q+1}$, $\bar{a}_r(t^\frown 0) = \bar{a}^0_q(t)$, $\bar{a}_r(t^\frown 1) = \bar{a}^1_q(t)$ for all $t \in 2^{n_q}$, $\bar{b}_r(t^\frown 0) = \bar{b}^0_q(t)$, $\bar{b}_r(t^\frown 1) = \bar{b}^1_q(t)$ for all $t \in 2^{n_q-k-1}$ and all $k < n_q$, and $\bar{b}_r(0) = z$ for some $z \in \omega^\omega$ such that $g(z) = \langle f(\bar{a}^0_q(s_{n_q})), f(\bar{a}^1_q(s_{n_q}))\rangle$. This latter demand can be fulfilled since by the assumptions, the $f(\bar{a}^0_q(s_{n_q})$ is G-connected to $f(\bar{a}^1_q(s_{n_q})$. Finally, let $r = \langle a_r, b_r\rangle$ be an approximation obtained from $\langle \bar{a}_r, \bar{b}_r\rangle$ by restricting all outputs of the functions $\bar{a}_r, \bar{b}_r$ to n_q+1. Then $r \leq q$ is an approximation validated by $\langle \bar{a}_r, \bar{b}_r\rangle$. This shows that q is not Y-terminal.
Case 2. Suppose that $A(q, Y)$ contains no G-edge for any one step extension $q \leq p$. Then each $A(q, Y)$ is an analytic G-anticlique, and it can be covered by a Borel G-anticlique $B(q, Y)$ by Claim 0.6. There are only countably many one-step extensions of p, and so these Borel anticliques can be enumerated by $\{B_n\colon n \in \omega\}$. We claim that this collection of anticliques works as in the claim.

To see this, let $Y' = Y \setminus \bigcup_n B_n$ and suppose that p is not Y'-terminal; i.e. there is a one step extension $q \leq p$ with a validation $\langle \bar{a}_q, \bar{b}_q\rangle$ such that $\mathrm{rng}(f \circ \bar{a}_p) \subset Y'$. Now, $f(\bar{a}_p(s_{n_q}) \in A(q, Y) \subset B(q, Y)$ by the definition of $A(q, Y)$, and at the same time $\mathrm{rng}(f \circ \bar{a}_p) \subset Y'$ and $Y' \cap B(q, Y) = 0$ by the construction of the set Y'. This is a contradiction proving the claim. □

Now, by transfinite recursion on an ordinal α, build countable sets C_α of Borel anticliques such that

- $C_0 = 0$ and $\alpha \in \beta \to C_\alpha \subset C_\beta$;
- writing $Y_\alpha = X \setminus \bigcup C_\alpha$, whenever p is an approximation all of whose extensions are Y_α-terminal, then p is $Y_{\alpha+1}$-terminal.

This is easy to do using Claim 0.8. Note that the sets $P_\alpha = \{p \colon p$ is an approximation which is not Y_α-terminal$\}$ are decreasing with α. Since the set of all approximations is countable, there must be a countable ordinal α such that $P_\alpha = P_{\alpha+1}$. There are two cases:
Case 1. The empty approximation belongs to P_α. In such a case, by induction on n build a sequence of approximations $0 = p_0 \geq p_1 \geq p_2 \geq \dots$ in P_α, using the fact that $P_\alpha = P_{\alpha+1}$ and so every approximation in P_α has a one step extension in $P_{\alpha+1}$. In the end, use Claim 0.7 to construct a continuous homomorphism h of G_0 to G.
Case 2. The empty approximation does not belong to P_α. In this case, the space X is covered by the anticliques in the set C_α. The theorem follows. □

We will now consider, without proof, several powerful generalizations of the G_0 dichotomy.

Definition 0.9. Let $m \in \omega$ be a natural number greater than 1. Let $Y = m^\omega$ be equipped with the product topology. Let $\{s_n \colon n \in \omega\} \subset m^{<\omega}$ be a dense collection of strings such that $s_n \in m^n$. Let G_m be the hypergraph of dimension n on Y defined by $G_m = \{\langle s_n^s mallfrowni^\frown z \colon i \in m\rangle \colon n \in \omega, z \in m^\omega\}$.

Theorem 0.10. *Let $m \in \omega$ be a natural number greater than* 1.

1. *The chromatic number of G_m is two;*
2. *the Borel chromatic number of G_m is uncountable;*
3. *whenever G is an analytic hypergraph of dimension ω on a Polish space X, then exactly one of the following occurs: either the Borel chromatic number of G is countable, or there is a continuous homomorphism $h \colon Y \to X$ of G_m to G.*

Definition 0.11. Let $\{s_n \colon n \in \omega\} \subset \omega^{<\omega}$ be a dense collection of strings such that $s_n \in m^n$. Let G_ω be the hypergraph of dimension ω on Y defined by $G_\omega = \{\langle s_n^s mallfrowni^\frown z \colon i \in \omega\rangle \colon n \in \omega, z \in \omega^\omega\}$. Let $Y \subset \omega^\omega$ be the G_δ set $\{z \in \omega^\omega \colon \forall m \exists k > m \forall i < k\ z(i) < k\}$.

Theorem 0.12. *1. The chromatic number of G_ω is two;*

2. *the Borel chromatic number of G_ω is uncountable;*
3. *whenever G is an analytic hypergraph of dimension ω on a Polish space X, then exactly one of the following occurs: either the Borel chromatic number of G is countable, or there is a continuous homomorphism $h \colon Y \to X$ of G_ω to G.*

1 Applications

We will now apply the graph dichotomy theorems to prove a number of more intuitive dichotomy results. The first two applications rely on a simple observation:

Proposition 1.1. *For every $x \in 2^\omega$, the connected component $[x]$ of G_0 containing x is equal to $\{y \in 2^\omega \colon \{m \in \omega \colon x(m) \neq y(m)\}$ is finite$\}$.*

Proof. The left-to-right inclusion is clear since two G_0-connected points differ at exactly one entry. For the right-to-left inclusion, for every number $n \in \omega$ consider the graph H_n on 2^n consisting of all pairs of the form $\{s_m^\frown 0^\frown t, s_m^\frown 1^\frown t\}$ for $m < n$ and $t \in 2^{n-m-1}$. We will prove that each graph H_n is connected.

This is proved by induction on n. The case $n = 0$ is trivial. Suppose now that H_n is connected and argue that H_{n+1} must be connected as well. Let $t_0, t_1 \in 2^{n+1}$ be arbitrary binary strings and look for a H_{n+1}-path between the two. If the last bit of t_0, t_1 is the same, then find an H_n-path from $t_0 \restriction n$ to $t_1 \restriction n$ and extend each string on the path by this bit. By the definitions, this yields an H_{n+1}-path from t_0 to t_1. If the last bits of t_0, t_1 are distinct, first use the induction hypothesis to find an H_n-path from $t_0 \restriction n$ to s_n and then one to s_n to $t_1 \restriction n$. Then, extend the strings on the first path by the last bit of t_0 and the strings on the second path by the last bit of t_1. The two paths taken together form an H_{n+1}-path from t_0 to t_1 as desired.

Now, if $x, y \in 2^\omega$ are points such that the set $\{m \in \omega \colon x(m) \neq y(m)\}$ is finite, find $n \in \omega$ larger than all elements of this set, find an H_n-path from $x \restriction n$ to $y \restriction n$ and append to its elements the sequence $x \restriction (\omega \setminus n) = y \restriction (\omega \setminus n)$. This yields a G_0-path from x to y as desired for the right-to-left inclusion. $\square$

Corollary 1.2. *If $C \subset 2^\omega$ is a closed set which, with each of its elements contains also all its G_0-neighbors, then $C = 0$ or $C = 2^\omega$.*

Proof. By the previous proposition, the G_0-components are dense. If C is nonempty, it must contain at least one whole dense G_0-component and since C is closed, $C = 2^\omega$. $\square$

Theorem 1.3. (Perfect set theorem) *Let $A \subset X$ be an analytic subset of a Polish space. Then either A is countable or A contains a nonempty perfect subset.*

Proof. Consider the graph G on X connecting points x, y just in case they are distinct elements of A. The graph G is analytic, and the G_0 dichotomy gives two options.

Either, the Borel chromatic number of G is countable. Since every G-anticlique contains at most one element of A, in this case the set A must be countable. Or, there is a continuous homomorphism $h \colon 2^\omega \to X$ of G_0 to G. The range of h, as a continuous image of a compact space, is compact. It is also a subset of A, since every point of 2^ω does have some neighbors in G_0 and the points in $X \setminus A$ do not have neighbors in G and the function h is a homomorphism of graphs. Thus, by the Cantor–Bendixson theorem, it will be enough to show that $\mathrm{rng}(h)$ is uncountable. To see this, note that for every point $x \in X$ the set $h^{-1}\{x\} \subset 2^\omega$ is a closed G_0-anticlique since h is a continuous homomorphism. Since countably many closed G_0-anticliques cannot cover 2^ω, it follows that $\mathrm{rng}(h)$ is uncountable as required. $\square$

Theorem 1.4. (Lusin–Novikov theorem) *Let $B \subset X \times Y$ be a Borel subset of a product of two Polish spaces. Then either B is the union of countably many Borel (partial) functions from X to Y, or there is a vertical section of B which contains a perfect subset.*

Proof. Let G be the graph on $X \times Y$ connecting two points just in case they are distinct elements of the same vertical section of the set B. The graph G is even Borel, and so the G_0 dichotomy gives us two options.

Either, the Borel chromatic number of G is countable. Since the intersection of every G-anticlique with the set B is a function, in this case the set B is covered by countably many functions. Or, there is a continuous homomorphism of $h\colon 2^\omega \to X \times Y$ of G_0 to G. First, observe that the range of h must be contained in a single vertical section of $X \times Y$. For this, note that the h-preimages of the vertical sections are closed by the continuity of h and apply Corollary 1.2. The rest of the argument is literally the same as in the perfect set theorem proof. $\square$

For the next two applications, we will need an old anticlique existence theorem due to Mycielski. It says that every small (meager) graph has a perfect anticlique.

Proposition 1.5. (Mycielski Theorem) *Let X be a Polish space without isolated points and G a meager graph on X. Then there is a nonempty perfect G-anticlique.*

Proof. Let O_n for $n \in \omega$ be open dense subsets of X^2 such that $G \cap \bigcap_n O_n = 0$. By induction on $n \in \omega$ build nonempty open sets P_t for all $t \in 2^n$ so that

- $t \subset s$ implies $\bar{P}_s \subset P_t$, and if t is incomparable with s then $P_s \cap P_t = 0$;
- the diameter of P_t is smaller than $2^{-|t|}$ for some fixed complete metric on X;
- if $t \neq s$ are distinct elements of 2^n then $P_s \times P_t \subset \bigcap_{m \in n} O_m$.

In the end, define a map $h\colon 2^\omega \to X$ by $h(z) =$the unique element of the intersection $\bigcap_n P_{z \restriction n}$. The image of h is a perfect subset of X; we must verify that it is an anticlique. Indeed, if $y \neq z$ are distinct elements of 2^ω and $m \in \omega$ is a number, find an $n \in \omega$ which is greater than m and such that $y \restriction n \neq z \restriction n$. Then, $h(y) \in P_{y \restriction n}$, $h(z) \in P_{z \restriction n}$, and $P_{y \restriction n} \times P_{z \restriction n} \subset O_m$. It follows that the pair $\langle h(y), h(z)\rangle \in X \times X$ belongs to the open set O_m for every $m \in \omega$ and so it is not in the graph G as desired. $\square$

The next proposition is going to be used to conclude that various graphs are meager. It ascertains that every non-meager graphon 2^ω with the Baire property contains a certain four point pattern in it. A G_0-*rectangle* is a set $\{y_0, y_1\} \times \{z_0, z_1\} \subset 2^\omega \times 2^\omega$ where $y_0, y_1, z_0, z_1 \in 2^\omega$ are points such that $y_0\ G_0\ y_1$ and $z_0\ G_0\ z_1$ holds. The points y_0, y_1, z_0, z_1 (in this order) will be called the *corners* of the rectangle

Proposition 1.6. *Let G be a graph on 2^ω with the Baire property. If G is not meager then G contains a G_0-rectangle.*

Proof. Since the graph $G \subset 2^\omega \times 2^\omega$ has the Baire property, it is equal to an open set modulo a meager set. Since G is not meager, that open set is nonempty. This means that there are binary strings $t, u \in 2^{<\omega}$ and sets O_n for $n \in \omega$ such that $O_n \subset [t] \times [u]$ is open dense in $[t] \times [u]$ and $\bigcap_n O_n \subset G$. Find incompatible binary strings t_0, u_0 extending t, u such that each of them is on the list $\{s_n : n \in \omega\}$ defining the graph G_0. For every binary string v write $(t_0^\frown 0^\frown v)' = t_0^\frown 1^\frown v$ and similarly

Now, by induction on $n > 0$ build binary strings t_n, u_n so that

- $t_0^\frown 0 \subset t_1 \subset t_2 \subset \ldots$ and $u_0^\frown 0 \subset u_1 \subset u_2 \subset \ldots$
- the four open sets $[t_{n+1}] \times [u_{n+1}]$, $[t_{n+1}] \times [u'_{n+1}]$, $[t'_{n+1}] \times [u_{n+1}]$, and $[t'_{n+1}] \times [u'_{n+1}]$ are all subsets of O_n.

To perform the induction step and get t_{n+1} and u_{n+1} from t_n, u_n, just extend the binary strings t_n, u_n repeatedly four times to handle each of the four sets in the fourth item. Each time use the assumption that $O_n \subset [t] \times [u]$ is an open dense set.

In the end, let $x_0 = \bigcup_n t_n$, $x_1 = \bigcup_n t'_n$, $y_0 = \bigcup_n u_n$ and $y_1 = \bigcup_n u'_n$. By the choice of the strings t_0, u_0, it is the case that $x_0\ G_0\ x_1$ and $y_0\ G_0\ y_1$. By the inductive construction, the G_0-rectangle $\{x_0, x_1\} \times \{y_0, y_1\}$ is a subset of $\bigcap_n O_n$ and therefore a subset of the graph G as required. $\square$

Theorem 1.7. (Silver's theorem) *Let E be a coanalytic equivalence relation on a Polish space X. Then either E has countably many classes or there is a perfect set of pairwise E-unrelated elements.*

Proof. Let G be the complement of E; this is an analytic graph. The G_0 dichotomy gives us two possibilities.

Either, the graph G has countable Borel chromatic number. In such a case, note that every G-anticlique is a subset of a single E-class, and therefore there are only countably many E-classes. Or, there is a continuous homomorphism $h\colon 2^\omega \to X$ of G_0 to G. The h-preimage F of E is an equivalence relation on 2^ω. It is coanalytic, and therefore has the Baire property. Note that F cannot contain a G_0 rectangle: writing y_0, y_1, z_0, z_1 for corners of such a rectangle, it would be the case that $h(y_0)\ E\ h(z_0)\ E\ h(z_1)$, by the transitivity of the equivalence relation $E\ h(y_0)\ E\ h(y_1)$, and this contradicts the assumption that h is a homomorphism of G_0 to the complement of E. By Proposition 1.6, the graph F is meager, and by Proposition 1.5 there is a perfect F-anticlique $C \subset 2^\omega$, and then $h''C$ is a perfect set of pairwise E-unrelated elements. $\square$

For the next application, a *quasimetric* on a set X is a function $d\colon X^2 \to \mathbb{R}$ such that its vaues are non-negative, it is symmetric, and it satisfies the triangle inequality. Unlike with a metric, the set X can contain distict points with d-distance zero. As good examples, consider the usual metric on $[0,1]^2$ and then the metric on $[0,1]^2$ which assigns points in the same vertical section their usual Euclidean distance, and points in distinct sections distance 1. These two metrics are not isomorphic: the former is separable, while the latter is not. The easiest way to see that the latter metric is not separable is to note that $\{\langle 0, r\rangle \colon r \in [0,1]\}$ is an uncountable, in fact perfect, collection of points with pairwise distance ≥ 1. The following dichotomy shows that this is the only way of proving non-separability of a definable quasimetric.

Theorem 1.8. *Let X be a Polish space and d a quasimetric on it such that for every $\varepsilon > 0$, the set $d^{-1}[\varepsilon, \infty) \subset X \times X$ is analytic. Then either d is separable or there is $\varepsilon > 0$ and a perfect set of elements of X with pairwise distances greater than ε.*

Proof. For each $\varepsilon > 0$ let G_ε be the graph on X connecting x, y if $d(x,y) \geq \varepsilon$; this is an analytic graph. The G_0 dichotomy gives us two options.

Either, for every $\varepsilon > 0$, the Borel chromatic number of G_ε is countable. Note that every G_ε-anticlique is a set of points of pairwise distance $< \varepsilon$, and so in this case for every $\varepsilon > 0$ the space X can be covered by countably many ε-balls, for every $\varepsilon > 0$. This implies that the metric d is separable.

Or, there is a real $\varepsilon > 0$ an a continuous homomorphism $h\colon 2^\omega \to X$ of G_0 to G_ε. Let e be the quasimetric on 2^ω defined by $e(y,z) = d(h(y), h(z))$ and consider the graph H on 2^ω connecting points y, z if their e-distance is $\leq \varepsilon/2$. The graph H is coanalytic and therefore has the Baire property. It cannot contain a G_0-rectangle: if y_0, y_1, z_0, z_1 were corners of such a rectangle, then $d(h(y_0), h(z_0)) \leq \varepsilon/2$, $d(h(y_1, z_0)) \leq \varepsilon/2$ would hold. At the same time, $d(h(y_0), h(y_1)) > \varepsilon$ since h is a homomorphism of G_0 to G_ε. This contradicts the triangle inequality for the quasimetric d. By Proposition 1.6, the graph H is meager, and by Proposition 1.5 there is a perfect H-anticlique $C \subset 2^\omega$. Then $h''C$ is a perfect set of points of pairwise d-distance at least $\varepsilon/2$. This completes the proof. $\square$

The next application concerns linear quasiorders on Polish spaces. Here, a *quasiorder* on a set is a transitive relation $\leq$ which contains the diagonal. It is *linear* if for all x, y in its domain, either $x \leq y$ or $y \leq x$. To motivate the result, consider the linear ordering $\leq$ on $\mathbb{R}$ and the linear ordering $\prec$ on $\mathbb{R}^2$ defined by $\langle x_0, x_1\rangle \prec \langle y_0, y_1\rangle$ if $x_0 < x_1$ or $x_0 = x_1$ and $y_0 \leq y_1$. They are not isomorphic since $\leq$ has a countable dense set while $\prec$ does not. One way to see that $\prec$ does not have a countable dense set is to observe that $[\langle x, 0\rangle, \langle x, 1\rangle]$ for $x \in \mathbb{R}$ is an uncountable collection of pairwise disjoint intervals in $\prec$. The following dichotomy shows that this is in fact the only way of showing that a definable linear quasiorder is not separable.

Theorem 1.9. *Let X be a Polish space and $\leq$ an analytic linear quasiordering on X. Then either $\leq$ is separable or else there is a perfect collection of pairwise disjoint $\leq$-intervals.*

Proof. We will need a notational convention. Say that a pair $\langle x, y\rangle \in X^2$ determines a nontrivial interval in $\leq$ if either $x \leq y$ or $y \leq x$ fails; the closed interval determined by the pair will be the set $\{z \in X : x \leq z \leq y\}$ or $\{z \in X : y \leq z \leq x\}$ depending on whether $\leq y$ or $y \leq x$ fails. We will identify the pairs with the closed intervals that they determine.

Consider the graph G on $X \times X$ connecting pairs $\langle x_0, x_1\rangle$ and $\langle y_0, y_1\rangle$ if they determine nontrivial closed $\leq$-intervlsl and these intervals have empty intersection. An inspection reveals that this is an analytic graph. The G_0 dichotomy provides us with two options.

Either, the (Borel) chromatic number of the graph G is countable. In such a case, the nontrivial closed intervals can be organized into countably many sets B_n for $n \in \omega$ such that in each B_n, any two intervals have nonempty intersection. In this case, we will argue that the completion $\leq^*$ of the quasiordering $\leq$ is separable, which certainly means that $\leq$ is separable. For each interval $I \in B_n$ write I^* for the corresponding interval in $\leq^*$. Let x_n^* be the least upper bound of the left endpoints of intervals in B_n, and observe that $x_n^* \in \bigcap\{I^* : I \in B_n\}$. The set $\{x_n^* : n \in \omega\}$ is dense in $\leq^*$ since every nontrivial closed interval in $\leq^*$ contains one of the points x_n^*.

Or, there is a continuous map $h: 2^\omega \to X \times X$ which is a homomorphism of G_0 to G. Note that for every $z \in 2^\omega$ the pair $h(z)$ defines a nontrivial interval in $\leq$, since only such pairs have any neighbors in the graph G. We must find a perfect set $C \subset 2^\omega$ such that $h''C$ consists of pairs defining pairwise disjoint intervals. Let H be the graph on 2^ω which is the complement of the h-preimage of G. This is a coanalytic graph, and therefore has the Baire property. It cannot contain a G_0-rectangle; to see this, observe that if I, J are disjoint closed intervals and K, L are closed intervals intersecting both I, J in a nonempty set then K, L cannot be disjoint. Thus, the graph H is meager by Proposition 1.6, and so by Proposition 1.5, there is a perfect H-anticlique $C \subset 2^\omega$. Then, $h''C$ is the requested perfect collection of pairwise disjoint intervals. $\square$