0.1 Lecture 6 – Compact spaces

0.1.1 Review

Let's pause to see how our powers have grown just from the last two results. Here are some examples of new results accessible to us using the theorems we proved in Lecture 5.

Example 0.1.1.
Look at the irrationals $\mathbb{R} \setminus \mathbb{Q}$ with the topology inherited from $\mathbb{R}$. Descriptively, $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ set. Why? Well, $\sim (\mathbb{R} \setminus \mathbb{Q}) = \mathbb{Q}$, which is a countable set. So

$$(\mathbb{R} \setminus \mathbb{Q}) = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\}.$$

Then $\mathbb{R} \setminus \mathbb{Q}$ is Polish, by Theorem **??**, and homeomorphic to Baire space.

Example 0.1.2.
We can easily show using Theorem **??** that $[0,1]^\omega$ is Polish. We can show also that a set is Polish if and only if it is homeomorphic to a G_δ set of $[0,1]^\omega$.

0.1.2 Compact spaces

Before we prove the next theorem, let's stop and review some basic observations about compact spaces that will help us in the next few results.

1. A closed subset of a compact space is compact.
2. Every continuous function from a compact space to $\mathbb{R}$ attains its supremum.[Z1]
3. Every sequence in a metrizable compact space has a convergent subsequence.[Z2]

0.1.3 Another result about Polish spaces

Theorem 0.1.1.
If X is a compact Polish space, and Y is a Polish space, then $C(X, Y)$ is Polish.

Proof. Let d be a complete metric on Y. We will argue that $e(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$ is a complete metric, and generates the compact-open topology.

(Completeness of e)
Suppose $\langle f_n : n \in \omega \rangle$ is an e-Cauchy sequence. For each $x \in X$, the sequence $\langle f_n(x) \rangle$ is d-Cauchy, because the distance between them is less than the distance d. Since d is complete, $\langle f_n(x) : n \in \omega \rangle$ converges to some $g(x)$.

Now we must argue that g is the limit. To do this, we must prove that

1. g is a continuous function from $X \to Y$.
2. $e(f_n, g)$ tends to zero.

We prove the second, and leave the first to the reader. Observe that if $\langle y_n : n \in \omega \rangle$ is a converging sequence of points in Y with $diam(\langle y_n : n \in \omega \rangle) \leq \epsilon$, then clearly $d(z, y_0) \leq \epsilon$.

Given some $\epsilon > 0$, we must find some $n_\epsilon \in \omega$ such that $\forall n > n_\epsilon$, $e(f_n, g) \leq \epsilon$. Which is the same as finding $\sup\{d(f_n(x), g(x) : n \in \omega,\ x \in X\}$.

We are chasing around definitions to get what we want.

[Z1] "This statement contains an implication that the set is bounded."
[Z2] "It must be metrizable, since the Stone-Czech ω is compact, but has no compact subsequence."

So if we find such a thing, then $\forall x \in X$, $d(f_{n_0}(x), f_{n_1}(x)) \leq \epsilon$. Then for each $x \in X$ and each $m > n_\epsilon$, $d(f_n(x), g(x)) \leq \epsilon$. And then $\forall n > n_\epsilon$, $e(g(x), f_n(x)) \leq \epsilon$. $\diamond$

(e generates the compact-open topology)
Now we must show that e generates the compact-open topology. To do this, we have to again prove two separate results.

1. If $\mathcal{O} \subset C(X, Y)$ is a basic open set, and $f \in \mathcal{O}$, we need an $\epsilon > 0$ such that the e-ball around f of radius ϵ is a subset of $\mathcal{O}$.
2. If $f \in C(X, Y)$, and $\epsilon > 0$, we need a basic open set $\mathcal{O} \subset C(X, Y)$ such that $f \in \mathcal{O}$ and $\mathcal{O} \subset B_e(f, \epsilon)$.

To prove the first, Find $n \in \omega$ such that $\langle K_m : m \in n \rangle$ are compact subsets of X, and $\langle \mathcal{O}_m : m \in n \rangle$ are open subsets of Y.

Then $\mathcal{O} = \{g : \forall m < n \forall x \in K_m(g(x) \in \mathcal{O}_n)\}$ is our basic open set.

We claim that $\exists \delta > 0$ such that $\forall x \in K_0$, the $B_d(f(x), \delta) \subseteq \mathcal{O}_0$. We must show this using compactness.

To show, consider a function $h(x) = \frac{1}{d(f(x), \sim\mathcal{O}_0)}$. Notice that h is a continuous function from $K_0 \to \mathbb{R}$, since K_0 is compact. So there is a bound r. Let $\delta = \frac{1}{r}$. Now let $\epsilon = min\{d_m : m \in n\}$. $\diamond$

To prove the second, Class ended $\square$