0.1 Lecture 5 – Results about Polish spaces

0.1.1 Elementary results about Polish spaces (continued)

We are going to just re-prove both directions of the theorem that we didn't finish last time, even though we finished one direction last time.

Theorem 0.1.1.
Let X be Polish, and $B \subset X$ a set. Then the following are equivalent.

1. *B is Polish with the restricted topology.*
2. *B is G_δ in X.*

Proof.
$(2 \to 1)$
Let d be a complete metric on X. Let $e(x, y) = d(x, y) + |\frac{1}{dist(x,\sim B)} - \frac{1}{dist(y,\sim B)}|$. Then e is a metric, as we observed last time. But we must check for completeness.

Suppose $\langle x_n : n \in \omega\rangle$ is an e-Cauchy sequence. Since e is a sum of positive numbers, each must form a d-Cauchy sequence. So it has a d-limit, call it y. Why is $y \in B$ true? Look at $\frac{1}{dist(x,\sim B)}$. This is also a Cauchy sequence. So there must be a limit, call it r. Then $dist(y, \sim B) = \frac{1}{r}$. Then $y \notin \sim B$, and so $y \in B$.

To conclude, note that a subspace of a separable metrizable space is separable metrizable. $X \supseteq \{x_n : n \in \omega\}$, and $B \subseteq X$. If X is metrizable, let $\{y_{mn} : m, n \in \omega\}$ be points such that $d(x_n, y_m n) < \frac{1}{m}$, and $y_{mn} \in B$, if possible (there may not be any such). Argue that this set must be dense in B.

$(1 \to 2)$
Suppose first that B is dense. We will see at the end that this is a reasonable and harmless assumption. Let e be a computable metric on B generating its topology. For each $n \in \omega$, let $\mathcal{O}_n = \{x \in X : \exists P$ an open nbhd of x s.t. $diam_e(P \cap B) < \frac{1}{n}\}$.

We fix n and say that x is in a neighborhood if the intersection has small diameter. Observe

- $\mathcal{O}_n$ is open ($\mathcal{O}_n \subseteq X$). Why? It is the union of open sets.
- Show $B = \cap\mathcal{O}_n$. Because $x \in B$, $B(x, \frac{1}{2n})$, there is an open set in the topology of X.

For every $x \in B$, find an open set $\mathcal{O} \subseteq X$ such that $(\mathcal{O} \cap B) = B_e(x, \frac{1}{3n})$ (B_e is a subset of B with e-diameter less than $\frac{1}{n}$, so $\mathcal{O}$ witnesses $x \in \mathcal{O}_n$). So $B \subseteq \mathcal{O}_n$.

Then $(\cap\mathcal{O}_n) \subseteq B$. Why? Let $x \in \bigcap_{n\in\omega} \mathcal{O}_n$, and let P_n be neighborhoods witnessing $x \in \mathcal{O}_n$. Now find $x_n \in B \cap (\bigcap_{m\in n} P_m)$. P_m is an open set of some sort and B is dense, so $B \cap (\bigcap_{m\in n} P_m)$ is not empty.

Now observe that the sequence $\langle x_n : n \in \omega\rangle$ is Cauchy in e. The radius of P_m is getting smaller, so the sequence must be Cauchy. So since e is complete, there must be a limit.

The limit must be in B, because e is complete on B. We want to argue that the limit is x.

To show the limit is x, choose $B_d(x, \frac{1}{n})$ for some metric d. Then $x_n \in B \cap (\bigcap_{m\in n} P_m \bigcap B_d(x, \frac{1}{n}))$.

If B is not dense in X then let Y be the closure of B. B is not dense, but Y is an arbitrary Polish topological space, so we will look at $Y = \bar{B}$. Then since it is closed, it is G_δ, and so it is Polish. Then B is G_δ in Y, so $B = Y \cap \mathcal{O}_n$ for some G_δ subset of X. But Y is closed, so Y is G_δ. So B is G_δ.

□

Theorem 0.1.2.
If $\langle X_n : n \in \omega \rangle$ are Polish spaces, then so is the product $\prod_{n\in\omega} X_n$, when equipped with the product topology.

Proof.
(Completely metrizable)
If d_n are complete metrics on each X_n, let $d(x,y) = \sum_n d_n(x(n), y(n))$ be a metric on the product. The problem is, the product could be infinite. So we cut it off, so that $|d_n| < 2^{-n}$. Then check for completeness and that it generates the product topology.

(Separable)
Let $\langle Y_n : n \in \omega \rangle$ be countable dense sets in X_n. Let $Y = \prod_{n\in\omega} Y_n$. But the product of countably many infinite sets is uncountable. Dense, sure, but uncountable. So let $Y = \{x \in X : \text{ all but finitely many } n \in \omega,\ x(n) = z_n \in X, \text{ otherwise } x(n) \in Y_n\ \}$.

Why is this set dense? Y is countable, since every Y is given by a finite set of ω. So it is countable. Then why dense? (visual argument on board ensues). □

Theorem 0.1.3.
Every Polish space is homeomorphic to a G_δ subset of the Hilbert cube $[0,1]^\omega$.

Because of this last theorem, the operations of countable product and taking G_δ subspace generate every Polish space that there is. Still, the others are worth studying and have their own interesting effects, but it is worth remembering.