0.1 Lecture 4 – Polish spaces

0.1.1 Examples of Polish spaces

Proposition 0.1.1.
In a metrizable space, closed sets are G_δ. A G_δ set is a countable intersection of open sets $\bigcap_{n\in\omega}\mathcal{O}_n$ for some open sets $\mathcal{O}_n$.

Proof. Let $\langle X, d\rangle$ be the metric space, and $\mathcal{C} \subseteq X$ a closed set. Let $\mathcal{O}_n = \{x \in X : \exists y \in \mathcal{C} : d(x,y) < 2^{-n}\}$. Observe that each $\mathcal{O}_n$ is open. Observe that $\mathcal{C} = \bigcap_{n\in\omega}\mathcal{O}_n$. So if $x \in \mathcal{C}$, then for all n, $x \in \mathcal{O}_n$. If $x \in \bigcap_{n\in\omega}\mathcal{O}_n$, there exist points $y_n \in \mathcal{C}$ such that $d(x, y_n) < 2^{-n}$. Then the sequence $\langle y_n : n \in \omega\rangle$ converges to $x \in \mathcal{C}$. □

Example 0.1.1.
The following are Polish spaces.

- Cantor space
- Baire space
- Euclidean space
- Separable Banach spaces

0.1.2 Operations on Polish Spaces

Taking a G_δ subspace

There will be a theorem soon where we will show that a subset of Polish space is Polish if and only if the subset is G_δ.

Taking a countably infinite product

We must preserve countability or we will not maintain separability.

Spaces of continuous functions

If X is compact[Z1], and Y is Polish, then $C(X, Y)$, that is, the space of all continuous functions from X to Y, is Polish.

If X and Y are $\mathbb{R}$ or $\mathbb{C}$, then $C(X, Y)$ is an easy example of a Banach space.

Hyperspaces

If X is Polish, one can form $K(X)$, the space of compact non-empty subsets of X. One can also form $F(X)$, the space of non-empty closed subsets of X. These are called hyperspaces.[Z2]

Space of all Borel probability measures on X

Sometimes this is notated $M(X)$, for a Polish space X. This is the beginning of dynamical systems, where you look at self-homeomorphisms and things of that nature.

[Z1] "What if X is not compact? Well, we can look at the space of continuous functions with compact support, or whose limit at infinity is 0, and so on..."
[Z2] "It sounds like something very fancy."

More structural operations

If G is a Polish group, and $H \subset G$ is a closed subgroup, then we can form G/H. It turns out that the cosets here, G/H, can be equipped with a topology that makes them Polish.

0.1.3 Digression: metrizable spaces

We will need a few more tools for the coming proofs. If X is a metrizable space with a metric d, can you mess[1] with d to get the same topology? Yes. there are a few ways to do this.

To verify, you must check that for any d-ball, you can find an e-ball which is a subset of the d-ball, with the same center, and vice-versa.

- Cut d off at any positive constant. That is, define $e(x,y) = d(x,y)$ if $d(x,y) \leq \epsilon$ for some ϵ, and $e(x,y) = \epsilon$ otherwise. Then you can verify that e is a metric, and generates the same topology.
- We can generalize the previous item and apply any concave increasing continuous function f to d, if $f(0) = 0$. Examples would be $\sqrt{d}$, $log(d)$, etc. You should verify the triangle inequality for these to see they are still metrics.
- If $f : X \to \mathbb{R}$ is a continuous function, then define $e(x,y) = d(x,y) + |f(x) - f(y)|$. This gives you a different metric, it is like the sum of two metrics, really. It generates the same topology.
- If d_n is a countable collection of metrics generating the same topology, such that $|d_n| < 2^{-n}$, then $\sum d_n$ is again a metric, and generates the same topology.
  To see this, let m be so large that $\sum_{m \geq n} 2^{-m} \leq \frac{\epsilon}{2}$. For each $n < m$, look at $B_n(x, \frac{\epsilon}{2n})$. Let $\mathcal{O} = \bigcap_{n<m} B_n(x, \frac{\epsilon}{2n})$. Then $\mathcal{O} \subset B_e(x, \epsilon)$.

Verifying this uses the fact that f is continuous.

0.1.4 Elementary results about Polish spaces

Theorem 0.1.1.
Let X be Polish, and $B \subset X$ a set. Then the following are equivalent.

1. *B is Polish with the restricted topology.*
2. *B is G_δ in X.*

Proof. ($2 \to 1$) Assume that B is open. Clearly B is separable. But we don't know whether it is metrizable. The restriction of B to this topology generates the topology, but the metric is not complete.

Let d be the metric on X. Let $f : B \to \mathbb{R}$ be the function which is equal to the distance to the complement. (Since the complement is closed, we can look at the distance to the infimum of the complement).

f is a continuous nonzero function on B (it is nonzero because B is open).

Then look at the metric $e(x,y) = d(x,y) + |\frac{1}{f(x)} - \frac{1}{f(y)}|$. This generates the same topology, but now, it is complete. How do we know it is complete? Because as you start getting close to the border, the sequence stops being Cauchy.

If B is G_δ, then $B = \bigcap_{n \in \omega} \mathcal{O}_n$. So for each $\mathcal{O}_n$ there is a complete metric. Fix d_n at each n. Choose $d_n \leq 2^{-n}$ on each $\mathcal{O}_n$. Let d be the sum metric of the intersection. Check for completeness.

(If Cauchy in d, then Cauchy in d_n). □

The lecture ended here. Tune in next time for the conclusion of the proof.

[1] His words