0.1 Lecture 3 – Topological structures, metric spaces, metrizability

0.1.1 Topological structures

It is possible to put additional structure on a topological space. There are many examples, and they are commonly used in many areas of mathematics.

Example 0.1.1 (Topological group).
A topological group $\langle G, \cdot, ^{-1}, \tau \rangle$ is a group $\langle G, \cdot, ^{-1} \rangle$ together with a topology τ on G, such that the operations $\cdot$ and $^{-1}$ are continuous maps of $G \times G \to G$ and $G \to G$, respectively.

Example 0.1.2 (Topological vector space).
A topological vector space $\langle V, +, \cdot, \tau \rangle$ over $\mathbb{R}$ is a vector space $\langle V, +, \cdot \rangle$ over $\mathbb{R}$ together with a topology τ on V such that the operations $+$ and $\cdot$ are continuous maps of $V \times V \to V$ and $\mathbb{R} \times V \to V$, respectively.

In a similar way, we can define topological rings, topological fields, and so on. These structures will appear later in the course, so it is good to be aware of them now.

0.1.2 Metric spaces

All of the topological spaces studied here (in this course) are generated by a metric. What does it mean for a topology to be generated by a metric?

Definition (Metric).
Let X be a set. A function $d : X \times X \to \mathbb{R}_0^+$ is a metric if for all $x, y, z \in X$:

- $d(x, x) = 0$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$.

The topology generated by the metric d is the one generated by all open metric balls, sometimes called d-balls.

Definition (Metric topology).
Given a set X, the metric topology or the topology generated by a metric d is the one generated (whose sub-basis consists of) open metric balls

$$\{B_d(x, \epsilon) : x \in X,\ (\epsilon > 0) \in \mathbb{R}\ \},$$

where each open metric ball B_d is given by

$$B_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}.$$

Definition (Metric space).
A topological space $\langle X, \tau \rangle$ is called a metric space, sometimes notated $\langle X, d \rangle$, for d a metric, if it is a topological space generated by (induced by) the metric d.

Definition (Metrizability).
A topological space $\langle X, \tau \rangle$ is metrizable if there exists a metric d such that the topology induced by d is τ. Another way to state this is that a topological space $\langle X, \tau \rangle$ is metrizable if it is homeomorphic to a metric space.

What is the difference between general topological spaces and metric spaces? Why would we be interested in metric spaces in particular? Well, metrizable spaces have nice features.

This is a consequence of the triangle inequality.

1. Metrizable spaces are regular Hausdorff. That is, given $x \neq y \in X$, there are disjoint open sets $\mathcal{O}_x, \mathcal{O}_y$ such that $x \in \mathcal{O}_x$, $y \in \mathcal{O}_y$, and $\mathcal{O}_x \bigcap \mathcal{O}_y = \emptyset$.
2. Every point in a metrizable space has a countable basis of neighborhoods, its "neighborhood basis", i.e. $\{B(x, \epsilon) : \epsilon > 0,\ \epsilon \in \mathbb{Q}\}$.
3. There are many others

0.1.3 Metrizability theorems

These are theorems that will say "if you have a topological structure, there is a nice metric which can generate that topology."[Z1]

Theorem 0.1.1 (Urysohn Theorem).
For a topological space $\langle X, \tau \rangle$, the following are equivalent

1. *X is separable metrizable*
2. *X has a countable basis and is regular Hausdorff*

There will be a similar theorem for topological groups. Given some $\langle G, \cdot, ^{-1}, \tau \rangle$, you may want a metric d such that d generates τ, and $d(xy, xz) = d(y, z)$. This would be a metric which is invariant from the left. And you can do that from the right as well.

Proposition 0.1.1.
Let X be a set, τ a topology, and d a metric. Then d generates τ if and only if the following are satisfied

1. $\forall \mathcal{O} \in \tau\ \forall x \in \mathcal{O}\ \exists \epsilon > 0$ *such that* $B(x, \epsilon) \in \mathcal{O}$.
2. $\forall x \in X\ \forall \epsilon > 0\ \exists \mathcal{O} \in \tau$ *such that* $x \in \mathcal{O} \subseteq B(x, \epsilon)$.

Proof. To show that every open metric ball belongs to τ, consider $B(x, \epsilon)$. Given $y \in B(x, \epsilon)$, there is some $\delta > 0$ such that $B(y, \delta) \subseteq B(x, \epsilon)$. By condition 2, there will be some open set $\mathcal{O} \in \tau$ such that $y \in \mathcal{O}$ and $\mathcal{O} \subseteq B(x, \epsilon)$.

This needs to be checked and probably fixed.

Then show that the union of every open ball containing a point in $B(x, \epsilon)$ is $B(x, \epsilon)$. ☐

0.1.4 Metrizable spaces

Example 0.1.3 ($\mathbb{R}$).
The topology of the real numbers $\mathbb{R}$ can be generated by the Euclidean metric, the absolute value of the difference between two points. $d(x, y) = \sqrt{(x-y)^2} = |x - y|$.

Example 0.1.4 ($\mathbb{R}^n$).
In general, the topology of $\mathbb{R}^n$ can be generated by the Euclidean distance, generalized a bit. For points $a, b \in \mathbb{R}^n$ such that $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$, we have $d(a, b) = \sqrt{\sum_{i=1}^{n} (b_i - a_i)^2}$.

There is nothing that says the metrization has to be unique.[Z2] The topology of $\mathbb{R}^n$ can also be generated by the Manhattan distance, where for points $a, b \in \mathbb{R}^n$ such that $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$, we have $d(a, b) = \sum_{i=1}^{n} |a_i - b_i|$.

Example 0.1.5 (2^ω).
Cantor space is generated by the smallest difference metric, that is $d(x, y) = 2^{-\delta(x,y)}$, where $\delta(x, y) = min\{n : x(n) \neq y(n)\}$. I claim that the product topology is generated by this metric.

[Z1] "It is always interesting when you see a topological structure to wonder whether the topology is generated by a metric."

[Z2] "This is important because if a given topological space is metrizable, there is not necessarily one "canonical" choice of metric."

Proof. We can use the proposition above. Given a metric ball around a point x, we must show that there is some open set which is a subset of the ball, and which contains x. An open ball in this topological space is an initial segment of a binary sequence.

Then since each initial segment is finite, there are a finite number of intersections of the d-balls, since δ will only pick out some finite position. □

Clean this up, it is pretty slick when he does it.

We could also generate this topology by finding closed subsets of $\mathbb{R}^n$ which are homeomorphic to 2^ω and apply the Euclidean metric.

For example, use Cantor's middle third set. This is homeomorphic to 2^ω, and is a closed subset of $\mathbb{R}$. Or we could use "Sierpinski's carpet".[1]

And there are many others.

Example 0.1.6 **(ω^ω).**
Can also be metrized with the smallest distance metric.

0.1.5 Non-metrizable spaces

Example 0.1.7 **($\mathbb{R}^\mathbb{R}$).**
The set $\mathbb{R}^\mathbb{R}$ of all functions $f : \mathbb{R} \to \mathbb{R}$ is not metrizable. They have a topology of pointwise convergence, which is really the product topology. Here, points do not have countable bases, so the space is not metrizable.

Think about why $\mathbb{R}^\mathbb{R}$ is different from $\mathbb{R}$.

Example 0.1.8 **($\beta\omega$).**
The set $\beta\omega$, the Stone-Ĉzech compactification of ω, is not metrizable, for reasons that "get complicated".

0.1.6 Completely metrizable spaces

Definition (Complete metric).
A metric d is called complete if any Cauchy sequence converges (has a limit). That is, if for all $\langle x_n : n \in \omega \rangle$, for all $\epsilon > 0$, there is some n such that for all m and k greater than n, $d(x_m, x_k) \leq \epsilon$.

Mostly we are interested in complete metrics, because for example in differential equations, and approximation, you want to look at accumulation points in a space.

Definition (Completely metrizable space).
A space is completely metrizable if there is a complete metric generating its topology.

As an example of complete versus non-complete metrics, consider the set ω with the discrete toplogy. If we metrize it with the discrete metric, we get a complete metric. If we metrize it by setting, say $d(m, n) = \frac{1}{m} - \frac{1}{n}$ if $m < n$, then the metric is not complete.

The take-home is that we have several different metrics here, but only one topology. So we think of topology over metrics.

However, in both examples, we have successfully generated the discrete topology. Therefore it might be possible to metrize a space in both a complete and a non-complete way.

Definition (Polish space).
A Polish space is a separable, completely metrizable space.

[1] He hesitated a few times, something sounded wrong to him about that name, and he mentioned that maybe it is called something else. But he was correct.