0.1 Lecture 2 – Topological spaces, homeomorphism, topological properties

0.1.1 Topological spaces

Definition (Topological space).
A topological space is a pair $\langle X, \tau \rangle$, where X is a set, and τ is a collection of subsets of X, satisfying the following conditions:

1. $X, \emptyset \in \tau$
2. If $\langle x_n : n \in \omega \rangle \in \tau$, then $(\bigcap_{n\in\omega} x_n) \in \tau$ (closed under countable intersection).
3. If $\langle x_n : n \in I \rangle \in \tau$, then $(\bigcup_{n\in I} x_n) \in \tau$ (closed under arbitrary union).

Definition (Continuous function).
For $\langle X, \tau \rangle$ and $\langle Y, \sigma \rangle$ topological spaces, [Z1] a function $f : X \to Y$ is called continuous if pre-images of σ-open sets are τ-open. $(\forall B \subseteq X(B \in \sigma \Rightarrow f^{-1}(B) \in \tau))$.

Definition (Homeomorphic function).
A function f is a homeomorphism if f is bijective and both f and f^{-1} are continuous. If f is a homeomorphism, the two topological spaces which f associates are called homeomorphic.

The question of whether or not two topological spaces are homeomorphic is a very basic question and arises often. To show that they are homeomorphic, one must exhibit a homeomorphism between them. To show that they are not homeomorphic, you must develop some means of proving that they are not. This may involve showing that no homeomorphism can exist, and this is usually done by finding a structure in one which cannot be continuously mapped to a structure in the other, or that such a map would not preserve a certain property, etc.

Definition (Basis for a topology).
Let $\langle X, \tau \rangle$ be a topological space. Then $A \subseteq \tau$ is a basis for τ if every set in τ is a union of some sets in A.

Definition (Sub-basis for a topology).
Let $\langle X, \tau \rangle$ be a topological space. Then $A \subseteq \tau$ is a sub-basis if τ is the smallest (relative to set-theoretic inclusion) topology containing A as a subset. Equivalently, $A \subseteq \tau$ is a sub-basis if $\{\cap A' : A' \subset A \text{ is finite}\}$ is a basis for τ.

Topologies are generally identified by identifying a sub-basis. We say that "such-and-such sets are basic open," or "these are the basic open sets," meaning that the sets which form the sub-basis of the topology we're looking at have a given form. We must often ask ourselves when working in a general setting, "what are the basic opens here?"

Example 0.1.1.
$\mathcal{P}(X)$, the discrete topology on X. The basic open sets are the singleton subsets of X.[Z2]

Example 0.1.2.
Given a linear order on X, the order topology is generated by the basic open intervals $(x, y) = \{z \in X : x < z < y\}$. $\mathbb{R}$ has this topology, but so do other sets, such as ω_2.[Z3]

Example 0.1.3.
The set $\mathbb{R}^n$ has a topology generated by open balls of positive diameter.

[Z1] "Topological spaces are the most general context in which continuity really makes sense."
[Z2] "Equivalently, every point is isolated."
[Z3] "This is a particularly nasty linear ordered set."

Example 0.1.4.
On a circle, there is a topology generated by all open intervals (arcs).

Example 0.1.5.
The set $\beta\omega$, the Stone-Čech compactification of ω, has a topology generated by the set of all ultrafilters containing...[Z4]

0.1.2 Properties of topological spaces

Definition (Accumulation point).
Given a topological space $\langle X, \tau \rangle$, and $S \subseteq X$, we say that $x \in X$ is an accumulation point of S if for every basic open neighborhood of x, that is, $\{\forall \mathcal{O} \in \tau : x \in \mathcal{O}\}$, $\mathcal{O}$ contains some point of S which is not x.

Claim 0.1.1.
The set of all accumulation points of S is closed.

Proof. If $x \notin acc(S)$, this is witnessed by some basic open neighborhood of x. Then $\sim acc(S) = \{\cup \mathcal{O} : S \cap \mathcal{O} = \{x\}\}$, the union of all open sets which do not contain any element of S except for one (x). The union of all these witnesses is open. Then $\sim acc(S)$ is open. Then $acc(S)$ is closed. □

Definition (Dense set).
Given a topological space $\langle X, \tau \rangle$, a set $S \subset X$ is dense (in X) if it intersects every non-empty open set, i.e. every non-empty open set contains at least one element of S.

Definition (Separable set).
Given a topological space $\langle X, \tau \rangle$, X is separable if it contains a countable dense subset.

Example 0.1.6.
The canonical example of a separable space is $\mathbb{R}$ equipped with its "usual" topology, that is, basic open sets are open intervals on the line. In this space, we can see that $\mathbb{Q}$ is a countable subset of $\mathbb{R}$, and $\mathbb{Q}$ is dense in $\mathbb{R}$, since from analysis we know that any real number can be approximated arbitrarily closely by a rational number, i.e., any real number is a limit point of some sequence of rationals. Then for any open ball around a real number, there will be some rational number within that ball, and thus $\mathbb{Q}$ is dense in $\mathbb{R}$.

This fact is more familiar than it may at first appear, as it is the entire basis for the decimal number system, or numbering systems in any base, in which a real number is written as a (possibly infinite) series of rationals:

$$a_0.a_1a_2...a_n = \sum_{i=0}^{n} \frac{a_i}{10^i} \text{ (for } a_0 \in \mathbb{Z} \text{ and } 0 \leq a_i \leq 9,\ i = \{1, 2, ..., n\})$$

Example 0.1.7.
Referring back to the examples of topological spaces and their basic open sets, we can see that the other spaces derived from $\mathbb{R}$, e.g. the circle, the line, and the plane, are all separable as well, since they corresponding subsets of $\mathbb{Q}$ can be found that are dense in those spaces.

Example 0.1.8.
We can form a topological space using ω_1, the smallest uncountable ordinal, and the order topology. In this context, ω_1 is sometimes notated $[0, \omega_1)$ to emphasize that it consists of all ordinals smaller than ω_1.

[Z4] He trails off and waves the chalk in little circles in the air instead of continuing. "The point is that basic opens can be complicated."

Suppose we had a countable subset $S \subset \omega_1$. Then $\sup S < \omega_1$. If $\alpha = \sup S$, then S is a subset of the closed set $[0, \alpha]$, so the closure of S is a subset of $[0, \alpha]$, which is a proper subset of ω_1. Then S is not dense in ω_1. Therefore ω_1 is not separable.

Definition.
A topological space $\langle X, \tau \rangle$ is called compact if every open cover of X has a finite subcover.

Example 0.1.9.
The closed unit interval is compact.

Partition-style proof. Assume for the sake of contradiction that the closed unit interval is not compact. Let $\mathcal{C}$ be an open cover of $[0, 1]$. Divide $[0, 1]$ into sub-intervals $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$. Since a union of finite subcovers would be finite, it must be the case that at least one of these sub-intervals is also not compact (i.e. has no finite subcover).

Choose one of the half-intervals which is not compact, and call it $[a_1, b_1]$. We can apply the same reasoning here, and determine that one of the halves of $[a_1, b_1]$, call it $[a_2, b_2]$, is also not compact.

Consider the nested sequence of closed intervals $\langle [a_n, b_n] \rangle_{n=1}^{\infty}$, none of which is compact, i.e., finitely coverable by our original open cover $\mathcal{C}$. Consider the "length" of the intervals in this sequence. Since we are dividing by $\frac{1}{2}$ at each step, we can write the length as

$$\lim_{n\to\infty} b_n - a_n = \lim_{n\to\infty} \frac{1}{2^n}$$

Then the diameters of these intervals decreases monotonically to 0. Then $\langle [a_n, b_n] \rangle_{n=1}^{\infty}$ is a sequence of nested, closed, bounded, and non-empty intervals. Then by the Cantor Nested Interval Theorem,

$$(\bigcap_{n=1}^{\infty} [a_n, b_n]) \neq \emptyset.$$

In fact, as an immediate corollary to that theorem, if the length of the sequence goes to 0, we have that

$$(\bigcap_{n=1}^{\infty} [a_n, b_n]) = \{p\},$$

where p is an urelement/atom, a single point.

Since $p \in [0, 1]$, there must be an open interval $O \in \mathcal{C}$ with $p \in O$. Then there is a some $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset O$. Let m be a positive integer such that $\frac{1}{2^n} < \epsilon$. Then since $p \in [a_m, b_n]$, we must have

$$[a_n, b_n] \subset (p - \epsilon, p + \epsilon) \subset O$$

But we have just covered $[a_m, b_m]$ with a single open set from the open cover $\mathcal{C}$. This contradicts the fact that $[a_m, b_m]$ is not finitely coverable by $\mathcal{C}$.

Then $[0, 1]$ is finitely coverable by $\mathcal{C}$, and is hence compact.[Z5] □

Example 0.1.10.
The real line $\mathbb{R}$, with the usual topology (basic open sets are open intervals of the line) is not compact.

Proof. We will show that any compact subspace of $\mathbb{R}$ is bounded, and hence cannot cover $\mathbb{R}$.

[Z5] "Take the unit interval and choose some open cover. Divide it in half, and, well one of these is not compact, so you choose that one. Keep dividing, and then you get the intersection of that, and it's a singleton, so, well that has a finite open cover, and so you get the contradiction."

Let $\mathcal{C}$ be the set of all open ϵ-balls of $0 \in \mathbb{R}$. Then the union of these is $\mathbb{R}$, and so this union is an open cover for any subset S of $\mathbb{R}$.

Let $\mathcal{F} \subset \mathcal{C}$ be a finite subcover of $\mathcal{C}$ for S. The union of all intervals in $\mathcal{F}$ is the largest open ϵ-ball in $\mathcal{F}$, and since $\mathcal{F}$ covers S, it follows that $S \subset \bigcup \mathcal{F}$. Then S is bounded by the radius of that largest ball, and so every compact subspace of $\mathbb{R}$ is bounded. Since $\mathbb{R}$ is not bounded, it follows that $\mathbb{R}$ cannot be compact. □

0.1.3 Basic operations on topological spaces

Taking the subspace

Verify that τ_S is indeed a topology by verifying closure under finite intersection and arbitrary union.

For a topological space $\langle X, \tau \rangle$, and a set $S \subseteq X$, S can be turned into a subspace $\langle S, \tau_S \rangle$ of $\langle X, \tau \rangle$, by equipping it with a topology τ_S inherited from X in the following way:

$$\tau_S = \{\mathcal{O} \cap S : \mathcal{O} \in \tau\}.$$

This is called the subspace topology, relative topology, or induced topology. It is the coarsest topology on S such that the inclusion map $f : S \to X$ is continuous.

Taking the product

For an arbitrary index set I, and topological spaces $\{\langle X_i, \tau_i \rangle : i \in I\}$, if we take the (possibly infinite)[Z6] Cartesian product $\prod_{i \in I} X_i$, it turns out there is a natural topology that will turn the Cartesian product of the sets X_i into a topological space: the product topology $\prod_{i \in I} \tau_i$.

Consider a function $f : (\prod X_i) \to X_i$. This is called a canonical projection. The product topology is defined to be the coarsest topology (i.e. the one with the fewest open sets) for which every canonical projection is continuous.

The product topology is generated by sets of the form $f_i^{-1}(\mathcal{O}_i)$, where $i \in I$ and $\mathcal{O}_i$ is an open subset of X_i. These open sets are sometimes called open cylinders. Their intersections are cylinder sets. A subset of $\prod X_i$ is open if and only if it is a union of intersections of finitely many sets of the form $f_i^{-1}(\mathcal{O}_i)$.

Prove that 2^ω, ω^ω, $[0,1]^\omega$, and $\mathbb{R}^\omega$ are not homeomorphic to eachother.

Show that 2^ω is compact. Hint: because 2 is compact, product of compact is compact, etc.

Example 0.1.11 (Notable product spaces).

2^ω	(Cantor space)	Countably many copies of $\{0, 1\}$.
ω^ω	(Baire space)	Countably many copies of ω with the discrete topology.
$[0,1]^\omega$	(Hilbert cube)	Countably many copies of $[0, 1]$ with $\mathbb{R}$'s topology.
$\mathbb{R}^\omega$		Countably many copies of $\mathbb{R}$ with $\mathbb{R}$'s topology.

[Z6] If I is infinite, we may need the Axiom of Choice to show that the underlying set is non-empty.