0.1 Lecture 1 – Introduction and course objectives

0.1.1 Texts

The text will be *Classical Descriptive Set Theory* by Alexander Kechris. Alternative texts exist, but are not suitable as replacements for the Kechris text for a variety of reasons.

- *Descriptive Set Theory* by Yiannis Moschovakis[1]
- *Set Theory* by Thomas Jech [2]
- *Invariant Descriptive Set Theory* by Su Gao [3]

0.1.2 Course Outline

The course will be organized into 3 blocks, and at the end of each block there will be a take-home exam. There will be two weeks allotted for the completion of each exam[Z1].

In terms of requisite knowledge, it would be helpful to be familiar with topological spaces, groups, and measures, particularly Borel probability measures.[Z2] Otherwise there will not really be prerequisites to what I will be talking about.

0.1.3 What is Descriptive Set Theory?

DST is the study of "definable" sets in mathematical analysis. "Definable" sets in the context of DST means Borel analytic subsets of Polish spaces. We can use the Borel hierarchy to gauge the complexity of these subsets. The first task is to define the category of Polish spaces, and then persuade you that just about all of analysis takes place in Polish spaces, and that most objects of interest to mathematical analysis are in fact Polish spaces. We will show the various categories of objects that can be organized into Polish spaces.

Example 0.1.1.
An object in the category of compact metrizable spaces can be a Polish space. Let S be a compact metrizable space. We will show that

$$S \cong K([0,1]^{\mathbb{N}}),$$

where $[0,1]^{\mathbb{N}}$ is the Hilbert cube, and $K(X)$ denotes the space of all compact subsets of X equipped with the Vietoris topology. We will show that $[0,1]^{\mathbb{N}}$ is a Polish space, and that if X is Polish, $K(X)$ is Polish. Therefore we can show that every compact metrizable space S is homeomorphic to a Polish space.

Example 0.1.2.
Banach spaces will also be seen to be Polish.

We will study the category of Polish groups, which are a kind of topological group. If you have not encountered the notion of topological group before, it is a natural idea. A topological group (G, τ) is a group G together with a topology τ on G, such that the group's binary operation and inverse operation are continuous with respect to the topology.

We will study an interesting feature of DST, **dichotomies**, which are not accessible by other means.

[1] "Not as good as Kechris."
[2] Only certain sections cover DST, too general.
[3] This semester is Classical DST, Su Gao and Invariant DST comes next semester.

[Z1] "I like to give you two weekends to work on it."
[Z2] "Really you don't need to know much."

Example 0.1.3 (Sketch of a dichotomy-type theorem).
Either a Borel set is "simple", or it contains a canonical obstacle to being "simple".[Z3]

Theorem 0.1.1 (Perfect set theorem).
If $A \subseteq \mathbb{R}$ is analytic, then exactly one of these two will occur:

- *A is countable* [4]
- *A contains a non-empty perfect subset*

A perfect set is closed and has no isolated points. A basic result is that perfect sets cannot be countable. Then this theorem gives us a strict dichotomy about the cardinality of an analytic subset A. This theorem lets us show, without any additional conditions or knowledge of A, that A must either be countable, or uncountable.[5]

Another notion of simplicity could be as follows. Consider a graph $\langle X, G \rangle$ on a Borel set X. The chromatic number of a graph is the smallest number of anticliques that can cover the graph. This number is quite difficult to evaluate, even for finite graphs. We can use DST to ask what is the Borel chromatic number, that is, the smallest number of Borel anticliques.

0.1.4 Spring 2016

In the spring continuation, DST II, we will be looking at classification of analytic equivalence relations.

Example 0.1.4.
Are these two compact metrizable spaces homeomorphic, or not?

Example 0.1.5.
Are these two separable Banach spaces isomorphic, or not?

Example 0.1.6.
Are these two varieties homeomorphic, or not?

We want to know the answer to these questions, because then we can classify these objects up to the corresponding morphism. To do this, we can compare the objects by their complexity in a descriptive sense, and use this to get traction on these problems, even in a very general setting.

[4] Here, A being countable is the notion of "simplicity" referred to in Example 0.1.3.
[5] You can use this to show that e.g. all analytic sets satisfy the continuum hypothesis.

[Z3] "It is very surprising that theorems of this kind can be proven."